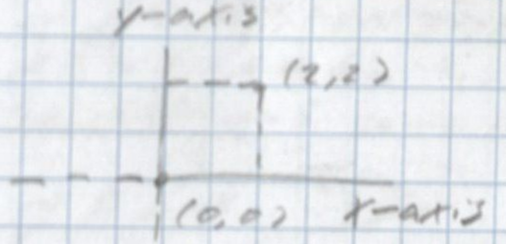
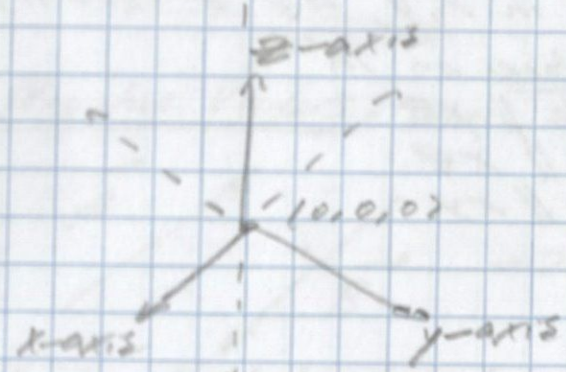
Calculus III

[Pre-Lecture Notes] 8.22.23

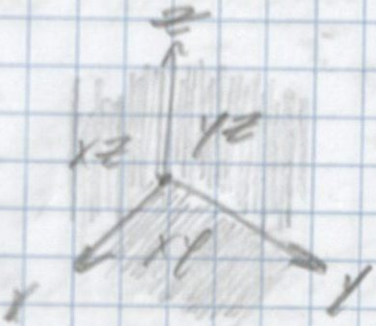
Introduction to 3D Space and Vectors:



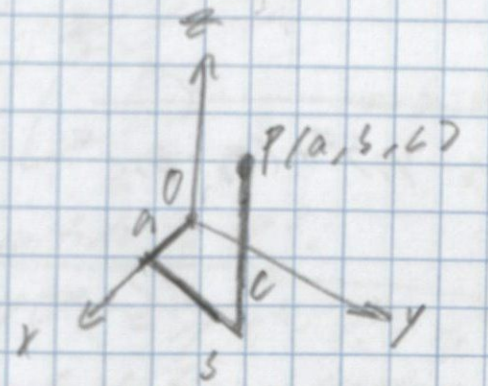
=> (a, b) coordinates on $\mathbb{R}^2$ representing (x, y)



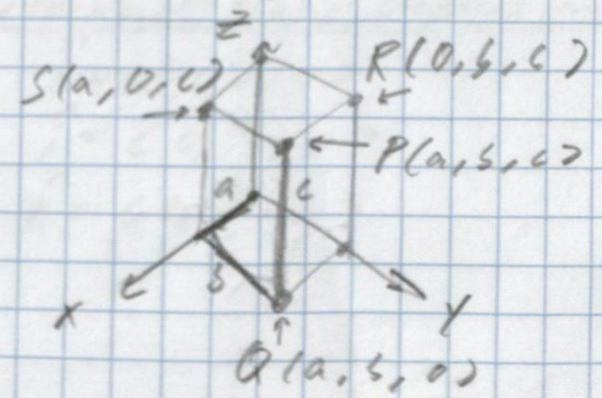
=> (a, b, c) coordinates on $\mathbb{R}^3$ representing (x, y, z)



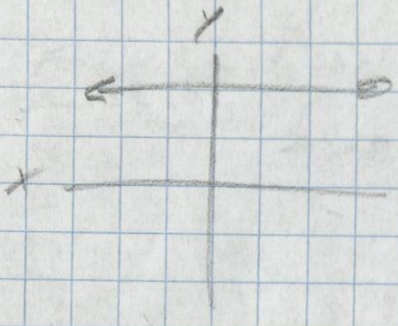
=> Three coordinate planes dividing $\mathbb{R}^3$ space into 8 octants. Shown is the foreground octant determined by the positive axis.



=> Point (a, b, c) in $\mathbb{R}^3$ space

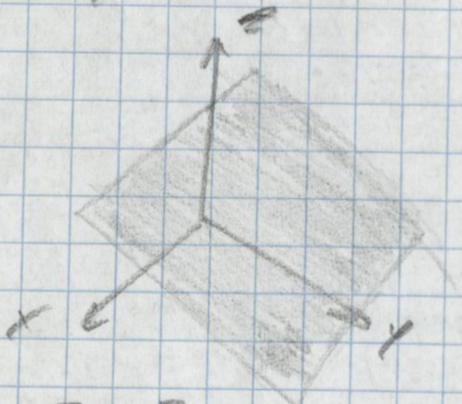


=> Projections of Q, R, and S of P onto the xy, yz, and xz planes of $\mathbb{R}^2$ respectively



$\Rightarrow$ equation in x and y results in a <u>curve</u> on $\mathbb{R}^2$

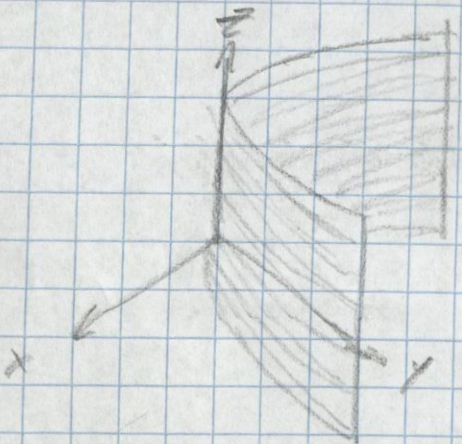
$y = 2$



$\Rightarrow$ equation in x, y, and z results in a <u>surface</u> on $\mathbb{R}^3$

- $x = k \parallel yz$
- $y = k \parallel xz$
- $z = k \parallel xy$

$z = 2$



$\Rightarrow y = x^2$ on $\mathbb{R}^3$

<u>Distance Formula in $\mathbb{R}^3$:</u>

$$* \; d(P_1, P_2) = |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

<u>Sphere Formula in $\mathbb{R}^3$:</u>

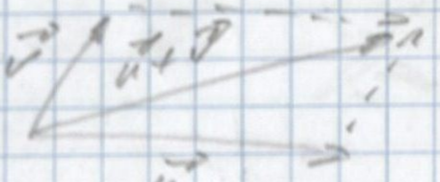
$$* \; r^2 = (x - h)^2 + (y - k)^2 + (z - l)^2$$

Vector: object with direction and magnitude

if $A=(a_1, a_2, a_3)$ and $B=(b_1, b_2, b_3)$

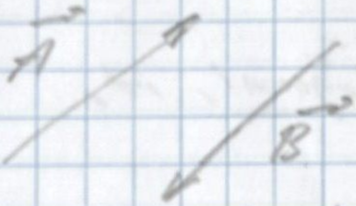
$\vec{AB} = B-A = \langle b_1-a_1, b_2-a_2, b_3-a_3 \rangle$

Vector Addition: same in $\mathbb{R}^3$



$\Rightarrow \vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2 \rangle$

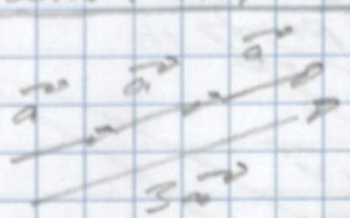
Negative Vectors: same in $\mathbb{R}^3$



opposite direction, same magnitude

$\Rightarrow \vec{A} = -\vec{B}$

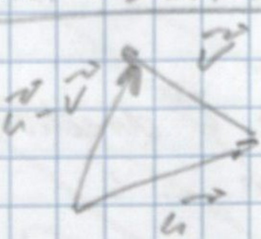
Scalar Multiplication: same in $\mathbb{R}^3$



scale magnitude of $\vec{v}$ by $|c|$,

$=$ same direction

Vector Subtraction: same in $\mathbb{R}^3$



$\Rightarrow \vec{u}-\vec{v} = \langle u_1-v_1, u_2-v_2 \rangle$

Magnitude in $\mathbb{R}^3$:

* $\|\vec{v}\| = \sqrt{v_1^2+v_2^2+v_3^2}$

Unit Vectors: vector of magnitude 1

* $\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$

Standard Basis Vectors in $\mathbb{R}^3$:

$\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$

Recovering Unit Vectors in $\mathbb{R}^2$:

* $\vec{v} = \|\vec{v}\|\cos\theta\,\hat{i} + \|\vec{v}\|\sin\theta\,\hat{j}$

The Dot and Cross Product:

Dot Product:

for $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

* $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$, also called a <u>scalar</u> or <u>inner product</u>

if the angle b/w $\vec{u}$ and $\vec{v}$ is θ,

* $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

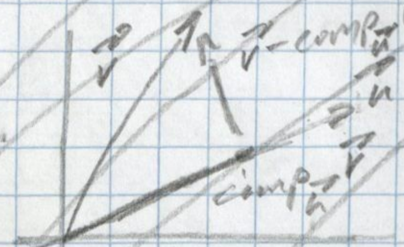
* $\Rightarrow \theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$

$\vec{u}$ and $\vec{v}$ are <u>perpendicular</u>, or <u>orthogonal</u>, if,

* $\vec{u} \cdot \vec{v} = 0$

Projections:

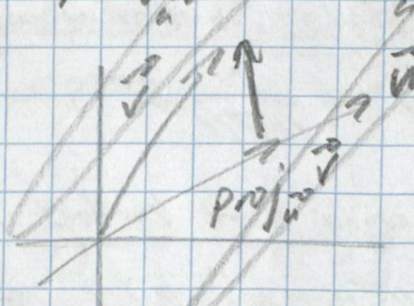
<u>Scalar Projections</u>: how much does $\vec{v}$ point in the direction of $\vec{u}$



$\Rightarrow \text{comp}_{\vec{u}} \vec{v} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$

<u>Vector Projections</u>: what piece of $\vec{v}$ is in the $\vec{u}$ direction?

$\text{proj}_{\vec{u}} \vec{v} = \text{comp}_{\vec{u}} \vec{v} \left(\dfrac{\vec{u}}{\|\vec{u}\|}\right) = \left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right) \vec{u}$



Cross Product:
given $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$,

$\vec{u}, \vec{v} \perp \vec{w} = \begin{cases} \vec{u} \cdot \vec{w} = 0 \\ \vec{v} \cdot \vec{w} = 0 \end{cases}$ is the cross product or vector product

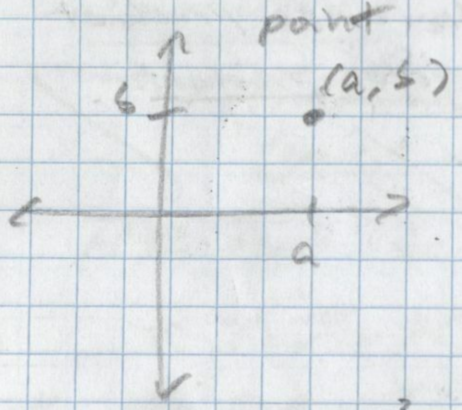
it uses determinates:
for 2x2

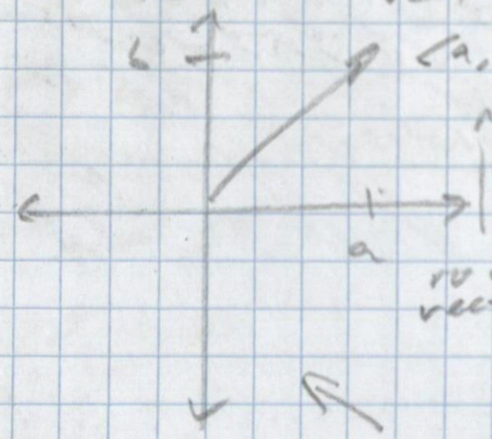
Lecture Notes

8.23.23

Points & Vectors

in $\mathbb{R}^2$:





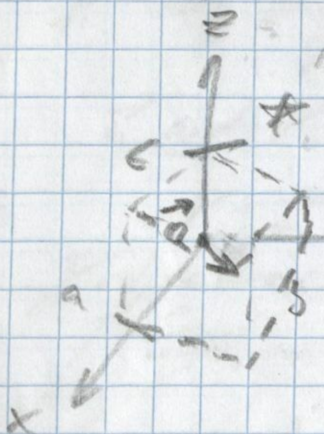
vector

$\langle a, b \rangle$ or $\begin{bmatrix} a \\ b \end{bmatrix}$

row vector   column vector

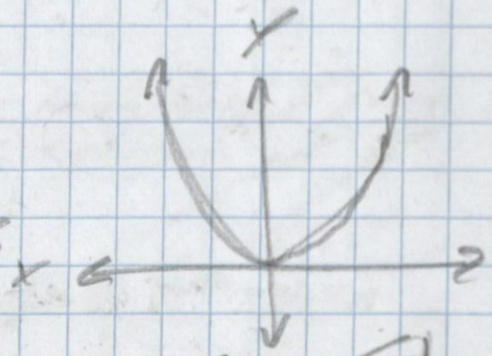
same object, different interpretation

in $\mathbb{R}^3$:



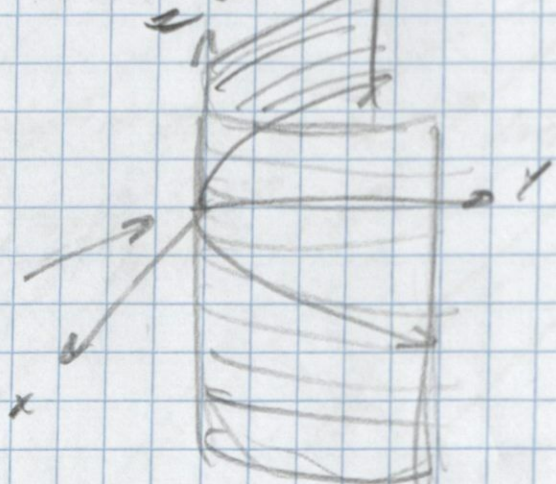
* $\langle a, b, c \rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

ex: $y = x^2$ in $\mathbb{R}^2$:



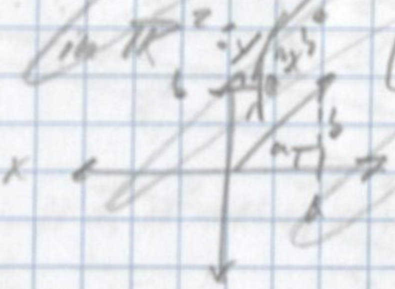
in $\mathbb{R}^3$:

* z can be anything

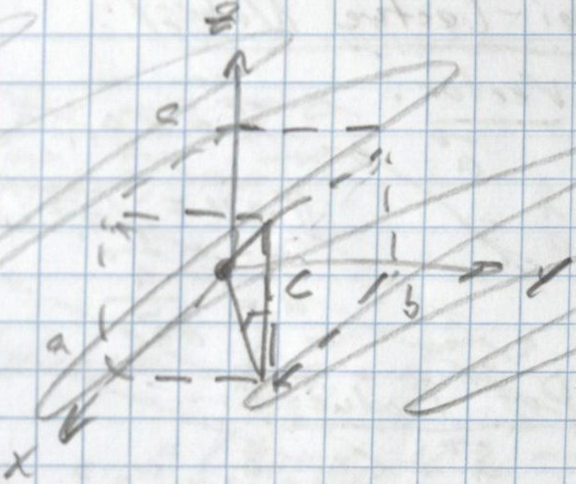


Distance:

in $\mathbb{R}^2$:



$\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^3$:



* Sphere:

radius r = distance from $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $\begin{bmatrix} h \\ k \\ l \end{bmatrix}$

ex- Show $x^2+y^2+z^2 = 4x-2y$ is a sphere

To complete the square

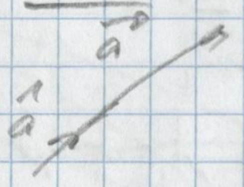
$$(x^2-4x)+(y^2+2y)+z^2=0$$
$$(x^2-4x+4-4)+(y^2+2y+1-1)+z^2=0$$
$$(x-2)^2+(y+1)^2+z^2=5$$

center: $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ radius: $\sqrt{5}$

Mini-Lecture Notes

8.24.23

Vector:



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|}$

Dot Product:

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\qquad\qquad$ $\vec{0} = \langle 0, 0, 0 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

* $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Remark:

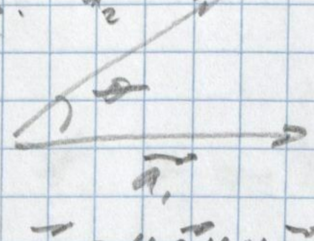
1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot \vec{0} = 0$
3. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$

   $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

Ex. $\vec{a} = \langle -1, 1, 2 \rangle$ $\quad$ $\vec{b} = \langle 0, 1, 1 \rangle$

$\vec{a} \cdot \vec{b} = 0 + 1 + 2 = 3$

Prob.



Proof:

* $\vec{a_1} \cdot \vec{a_2} = \|\vec{a_1}\| \|\vec{a_2}\| \cos\theta$

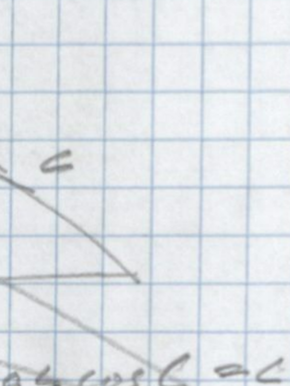
$\cos\theta = \dfrac{\vec{a_1} \cdot \vec{a_2}}{\|\vec{a_1}\| \|\vec{a_2}\|}$

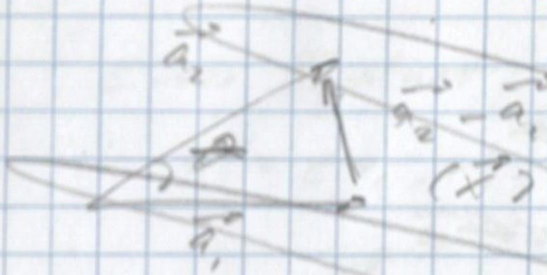
* $\theta = \cos^{-1}\left(\dfrac{\vec{a_1} \cdot \vec{a_2}}{\|\vec{a_1}\| \|\vec{a_2}\|} \right)$

$a^2 + b^2 - 2ab\cos C = c^2$

$C = \frac{\pi}{2}$ $\quad$ $\cos C = 0$

$a^2 + b^2 = c^2$





$\vec{a}_1 + \vec{x} - \vec{a}_2 = 0$

$\Rightarrow \vec{x} = \vec{a}_1 - \vec{a}_2$

$\|\vec{a}_1\|^2 + \|\vec{a}_2\|^2 - 2\|\vec{a}_1\|\|\vec{a}_2\|\cos\theta = \|\vec{a}_2 - \vec{a}_1\|^2$

$= (\vec{a}_2 - \vec{a}_1)(\vec{a}_2 - \vec{a}_1)$

$= \|\vec{a}_2\|^2 - \vec{a}_2\vec{a}_1 - \vec{a}_1\vec{a}_2 + \|\vec{a}_1\|$

Ex. $\vec{a} = \langle -1, 1, 2 \rangle \quad \vec{b} = \langle 0, 1, 1 \rangle$

$\theta = \cos^{-1}\left(\frac{\vec{a}_1 \cdot \vec{a}_2}{\|\vec{a}_1\|\|\vec{a}_2\|}\right)$

$= \cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{2}}\right)$

$= \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right)$

$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

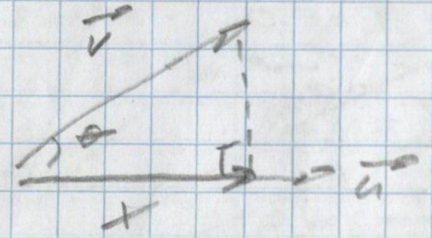
$= \frac{\pi}{6}$

$\sqrt{6}\cdot\sqrt{2}$

$\sqrt{3}\cdot\sqrt{2}\cdot\sqrt{2}$

$2\sqrt{3}$

Component & Projection:



$$\frac{x}{\|\vec{v}\|} = \cos\theta$$

$$x = \|\vec{v}\|\cos\theta$$

$$\text{comp}_{\vec{u}}\vec{v} = \|\vec{v}\|\cos\theta$$

* component of $\vec{v}$ in the direction of $\vec{u}$

$$= \frac{\|\vec{u}\|\|\vec{v}\|\cos\theta}{\|\vec{u}\|}$$

$$\ast \quad = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$$

$$\text{proj}_{\vec{u}}\vec{v} = \|\vec{v}\|\cos\theta\,\frac{\vec{u}}{\|\vec{u}\|}$$

$$\|\vec{a}\| = r$$

$$\vec{a} = r\cdot\frac{\vec{a}}{\|\vec{a}\|}$$

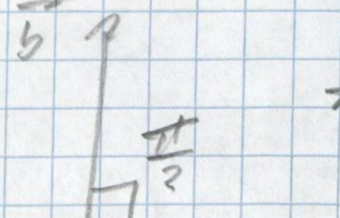
$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\cdot\frac{\vec{u}}{\|\vec{u}\|}$$

$\hat{u}$ (changes it from a scalar to an vector)

$$\ast \quad = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}$$

$$\ast \quad \|\text{proj}_{\vec{u}}\vec{v}\| = \text{comp}_{\vec{u}}\vec{v}$$

$\vec{a} \perp \vec{b}$



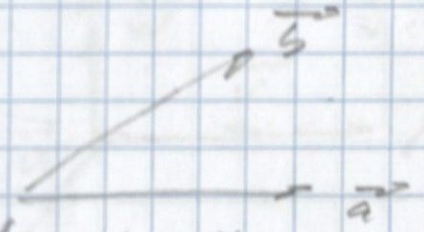
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\ast \quad \vec{a}\cdot\vec{b} = 0 \Longleftrightarrow \vec{a}\perp\vec{b}$$

$$\cos\frac{\pi}{2} = 0$$

Cross Product:

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$



$\vec{a} \cdot \vec{c} = 0$

$\vec{b} \cdot \vec{c} = 0$

$\vec{c} = \vec{a} \times \vec{b}$

determinant

$\vec{c}$ comes out of page

* $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}$$

$$+ a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_3 \end{bmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_3 - b_2 c_1)$$

$+ \vec{i} = \langle a_2 b_3 - a_3 b_2 \rangle$

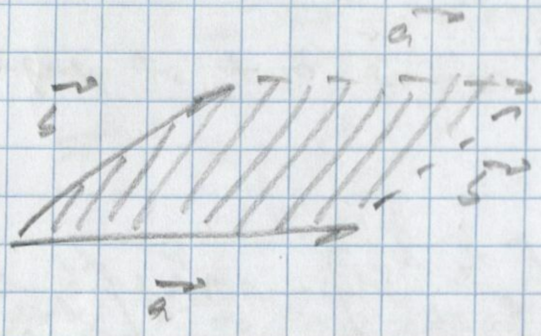
$- \vec{j} = \langle a_1 b_3 - a_3 b_1 \rangle$

$+ \vec{k} = \langle a_1 b_2 - a_2 b_1 \rangle$

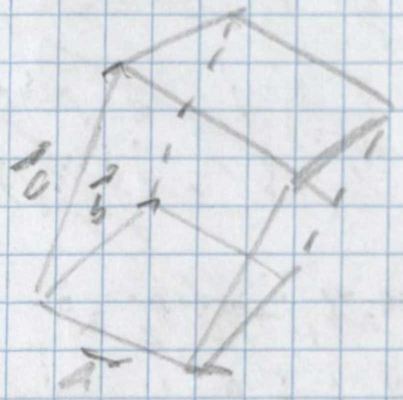
$\vec a \cdot \vec b = \vec b \cdot \vec a$ $\quad$ 1. $\vec a \times \vec b = -\vec b \times \vec a$

Prop.

1. $\|\vec a \times \vec b\| = \|\vec a\|\|\vec b\|\sin\theta$
2. $\vec a \times \vec b = \vec 0 \iff \vec a \parallel \vec b$



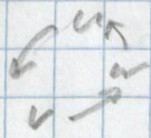
$A = \|\vec a \times \vec b\|$



Box Product:

$\vec u, \vec v, \vec w$

$= \vec u \cdot (\vec v \times \vec w)$



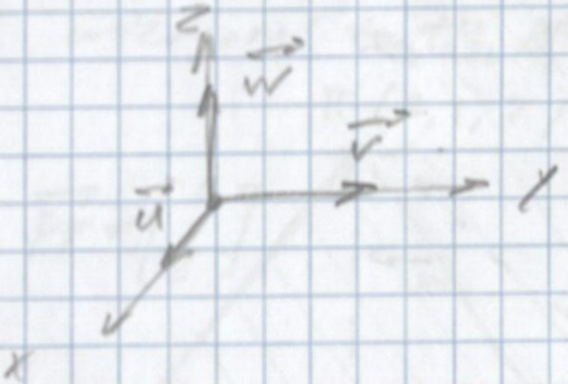
$= \vec v \cdot (\vec w \times \vec u)$

$= \vec w \cdot (\vec u \times \vec v)$

Independent Notes

8.24.23

Cross Product:



$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{w} = 0$

$\vec{v} \cdot \vec{w} = 0$

$\vec{w} = \vec{u} \times \vec{v}$

Determinants:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product Formula:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

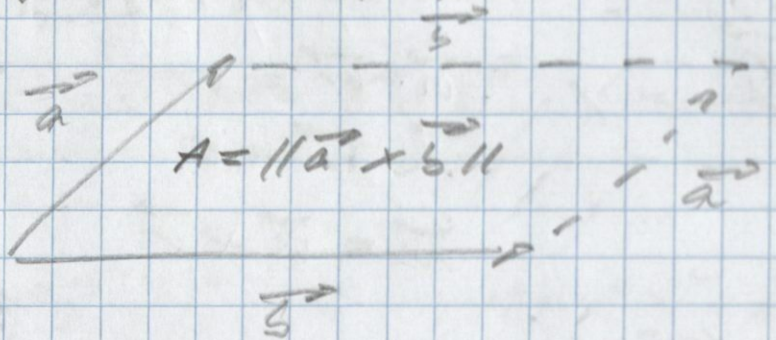
$$\vec{a} \times \vec{b} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - a_3 b_1) + \hat{k}(a_1 b_2 - a_2 b_1)$$

$$= \langle a_2 b_3 - a_3 b_2, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - a_2 b_1 \rangle$$

Properties of Cross Product:

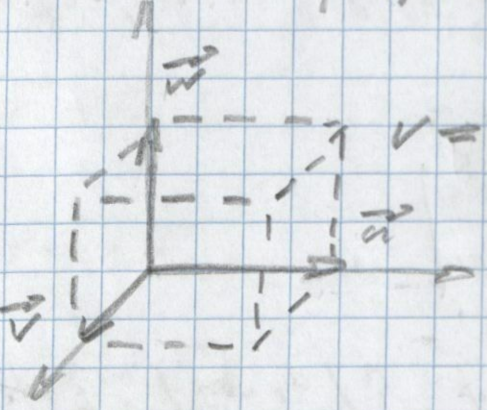
1. $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$
2. $\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \| \vec{b} \| \sin\theta$



3. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Box Product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

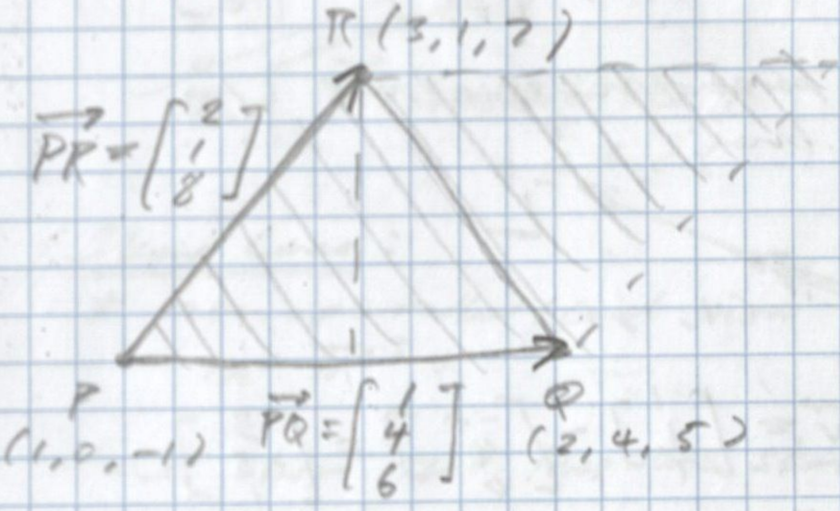


$V = \vec{u} \cdot (\vec{v} \times \vec{w})$

Lecture Notes

8.25.23

Find the area of triangle PQR and a vector orthogonal to the plane PQR.



vector orthogonal to △PQR

$$\begin{bmatrix} 6-32 \\ 8-12 \\ 8-1 \end{bmatrix} = \begin{bmatrix} -26 \\ -4 \\ 7 \end{bmatrix}$$

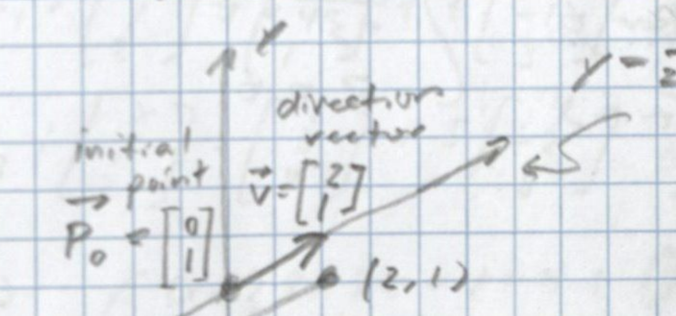
$$\text{Area of } \triangle = \frac{1}{2} \cdot (\text{Area of } \square) = \frac{1}{2} \left\| \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \right\|$$

$$= \frac{1}{2}\sqrt{(-26)^2 + (-4)^2 + 7^2} = \frac{1}{2}\sqrt{741}$$

Lines in $\mathbb{R}^2$:

$$y - y_0 = m(x - x_0)$$

$$y = mx + b$$

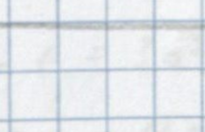


vector equation:

$$* \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

parameter form:

$$* \quad \begin{cases} x = 0 + 2t \\ y = 1 + t \end{cases}$$

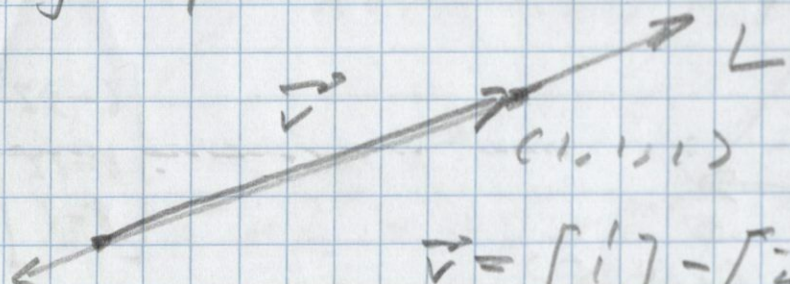
$\vec{v} = \begin{bmatrix} 1 \\ m \end{bmatrix} =$ 

Lines in $\mathbb{R}^3$:

$\vec{v} = \langle a, b, c \rangle$ vector form:

$\vec{P_0} = \langle x_0, y_0, z_0 \rangle$ $*$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

ex. find vector and parametric equations s/w given points on line



$P_0 = (-2, 4, 0)$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$

vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

parametric form:

$$\begin{cases} x = -2 + 3t \\ y = 4 - 3t \\ z = t \end{cases} \xrightarrow{\text{solve for } t} \begin{cases} \frac{x+2}{3} = t \\ \frac{y-4}{-3} = t \\ z = t \end{cases}$$

symmetric form: $(t =)$

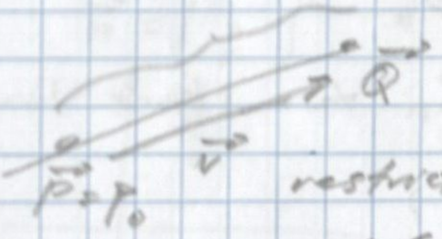
$$\frac{x+2}{3} = \frac{y-4}{-3} = z$$

$$* \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

ex. $\begin{cases} x = -2 \\ y = 4 - 3t \\ z = t \end{cases}$

$$\Rightarrow x = -2; \ \frac{y-4}{-3} = z$$

Line Segments:



restrict the t

$a \leq t \leq b$

Skew Lines:

* non-parallel and non-intersecting lines in $\mathbb{R}^3$ *

ex. Show that lines are skew.

$L_1: x = 3+2t,\ y = 4-t,\ z = 1+3t$

$L_2: x = 1+4s,\ 3-2s,\ z = 4+5s$

Check: (1) non-parallel and (2) non-intersecting

(direction vectors are scalar multiples of each other ($L_1 \times L_2 = 0$))

(1) $L_1: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$L_2: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + s \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$

direction vectors (not scalar multiples $\therefore$ not parallel)

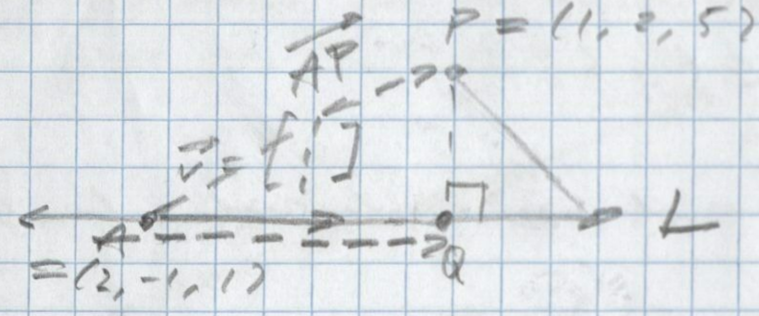
(2) $\begin{cases} (x=)\ 3+2t = 1+4s & ① \\ (y=)\ 4-t = 3-2s & ② \\ (z=)\ 1+3t = 4+5s & ③ \end{cases}$

Show no solutions for s, t

①: $2t = -2+4s \Rightarrow t = -1+2s$

②, ③: $\begin{cases} ②\ 4-(-1+\cancel{2s}) = 3-\cancel{2s} \\ ③\ 1+3(-1+2s) = 4+5s \end{cases}$ $5 \neq 3$ no solution ex. Let L be the line through $\begin{bmatrix}2\\-1\\1\end{bmatrix}$ in the direction $\begin{bmatrix}1\\1\\1\end{bmatrix}$. What is the point on L closest to $\begin{bmatrix}1\\2\\5\end{bmatrix}$?

ex. Find the distance between two skew lines



Method 1: minimize (squared) distance *

vector form: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2+t\\-1+t\\1+t\end{bmatrix}$

distance squared b/w $\begin{bmatrix}2+t\\-1+t\\1+t\end{bmatrix}$ and $\begin{bmatrix}1\\2\\5\end{bmatrix}$ is $\left\|\begin{bmatrix}1\\2\\5\end{bmatrix} - \begin{bmatrix}2+t\\-1+t\\1+t\end{bmatrix}\right\|^2$

$= \left\|\begin{bmatrix}-1-t\\3-t\\4-t\end{bmatrix}\right\|^2 = (-1-t)^2 + (3-t)^2 + (4-t)^2$ quadratic in t

Method 2: find Q such that $\vec{QP} \cdot \vec{v} = 0$

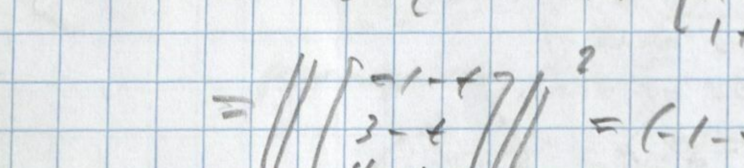
$* \ \vec{Q} = \begin{bmatrix}2+t\\-1+t\\1+t\end{bmatrix}$ $\qquad \begin{bmatrix}-1-t\\3-t\\4-t\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}$

$(-1-t)(1) + (3-t)(1) + (4-t)(1) = 0$

$-3t + 6 = 0 \Rightarrow t = 2$

$\vec{Q} = \boxed{\begin{bmatrix}4\\1\\3\end{bmatrix}}$

Method 3:

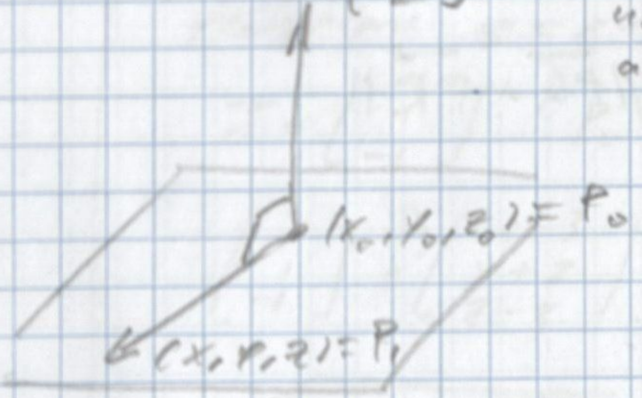


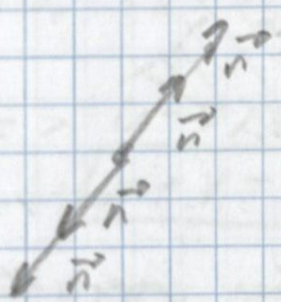
$\vec{AQ} = \text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \frac{\begin{bmatrix}-1\\3\\4\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{-1+3+4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$Q = A + \vec{AQ} = \begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}2\\2\\2\end{bmatrix} = \boxed{\begin{bmatrix}4\\1\\3\end{bmatrix}}$

[Lecture Notes] Planes 8.28.23

$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ * normal vector only unique up to multiplication by a nonzero scalar





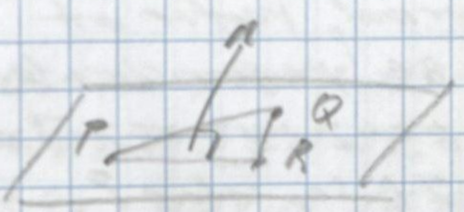
* $\vec{n} \cdot \overrightarrow{P_0 P_1} = 0$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$ * scalar equation of the plane *

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\Rightarrow ax + by + cz + d = 0$

ex. Find an equation for the plane passing through $P = (0, 1, 1)$, $Q = (1, 0, 1)$, and $R = (1, 1, 0)$



find $\vec{n}$ and point P_0

Take $P_0 = P = (0, 1, 1)$

Take $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 - 0 \\ 0 - (-1) \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Formula: $\vec{n} \cdot \overrightarrow{P_0 \text{ to } (x, y, z)} = 0$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x - 0 \\ y - 1 \\ z - 1 \end{bmatrix} = 0$$

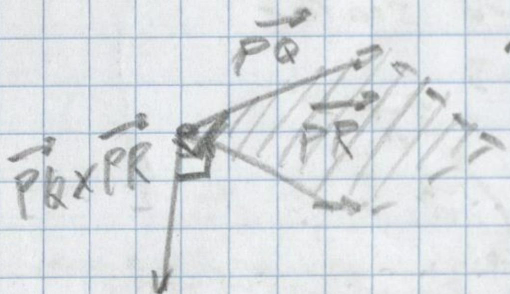
$x + (y - 1) + (z - 1) = 0 \Rightarrow x + y + z = 2$

Check: plug in $P=(0,1,1)$: $0+1+1=2$ ✓

$Q=(1,0,1)$: $1+0+1=2$ ✓

$R=(1,1,0)$: $1+1+0=2$ ✓

$\text{Area} = \|\vec{PQ} \times \vec{PR}\|$



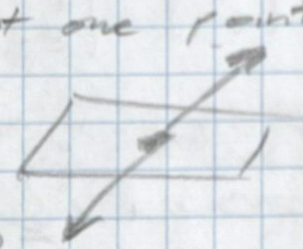
2 **Lines:** intersecting, parallel, skew

Line and Plane: parallel, intersecting

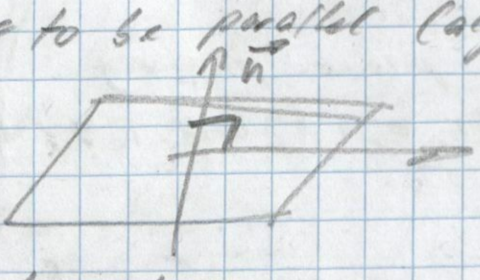
line contained in plane

line never intersects plane

at one point



What does it mean for a line and a plane to be parallel (algebraically)?



direction vector of line and normal vector of plane are perpendicular

Ex.

Does the line $L: x=3+\underline{3}t,\ y=\underline{t},\ z=-2+\underline{4}t \Rightarrow \vec{v}=\begin{bmatrix}3\\1\\4\end{bmatrix}$ direction vector

intersect the plane $\underline{x}+\underline{y}+\underline{z}=2$? If so, where?

1. Check whether parallel? $\vec{n}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ normal vector

$\vec{n}\cdot\vec{v} = 3+1+4 = 8 \neq 0$

∴ not parallel

∴ intersect at __unique__ point

2. To find intersection, substitute parametrization of L __into__ $x+y+z=2$

$(3+3t)+(t)+(-2+4t)=2$

$1+8t=2$

$t=\frac{1}{8} \longrightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3+3(\frac{1}{8})\\ \frac{1}{8}\\ -2+4(\frac{1}{8})\end{bmatrix} \boxed{= \begin{bmatrix}22/8\\1/8\\-3/8\end{bmatrix}}$

Lecture Notes | Planes 8.30.23

ex. Find the equation of the plane through the point $(3,-2,8)$ and parallel to the plane $z = x+y$

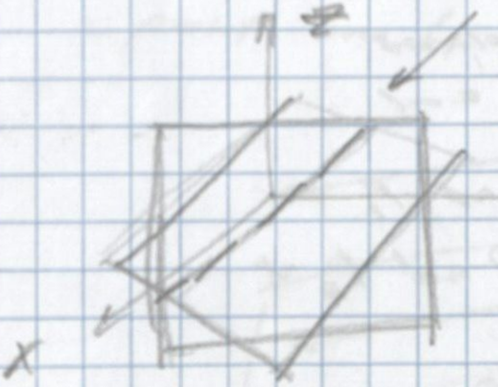
$$\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad P_0 = (3,-2,8)$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x-3 \\ y+2 \\ z-8 \end{bmatrix} \Rightarrow (x-3)+(y+2)-(z-8)=0$$

$$\boxed{x+y-z = -7}$$

2 planes intersect in a line

angles b/w planes = angle b/w normal vectors



$$\vec{n}_1 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \qquad \vec{n}_2 = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \therefore \underline{\text{not}} \text{ parallel}$$

ex. Do the planes $2x-3y+4z=5$ and $x+6y+4z=3$ intersect? If so, what is their angle of intersection? Also, what is the equation for the line of intersection?

(1) θ = angle of intersection = angle b/w $\vec{n}_1$ and $\vec{n}_2$

$$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\| \|\vec{n}_2\| \cos\theta$$

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \sqrt{4+9+16}\sqrt{1+36+16} \cos\theta$$

$$2-18+16 = \sqrt{29}\sqrt{53}\cos\theta \quad \therefore \theta = 90°$$

(2) Line determined by point + direction vector

$\downarrow \perp$ to $\vec{n}_1$ and $\vec{n}_2$

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -12-24 \\ 4-8 \\ 12+3 \end{bmatrix} = \begin{bmatrix} -36 \\ -4 \\ 15 \end{bmatrix}$

Find a point on both planes:

$\begin{cases} ① \; 2x - 3y + 4z = 5 \\ ② \; x + 6y + 4z = 3 \end{cases}$

set $x = 0$

* finds one sol as there are ∞ sol and only need 1 *

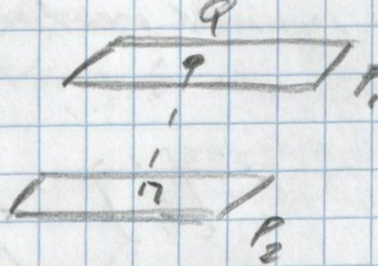
① − ② : $-9y = 2 \;\therefore\; y = -\frac{2}{9}$

Plug into ①: $-3(-\frac{2}{9}) + 4z = 5 \rightsquigarrow \frac{2}{3} + 4z = 5 \rightsquigarrow z = \frac{13}{12}$

$\therefore$ point $= \begin{bmatrix} 0 \\ -2/9 \\ 13/12 \end{bmatrix}$ line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2/9 \\ 13/12 \end{bmatrix} + t \begin{bmatrix} -36 \\ -4 \\ 15 \end{bmatrix}$

* rather than find $\vec{v}$, could also find two points on line *

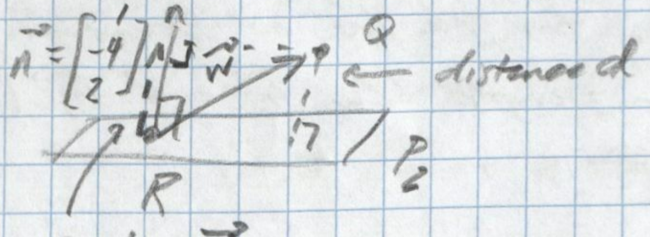
ex. Find the distance b/w the parallel planes
$x - 4y + 2z = 0$ and $2x - 8y + 4z = -1$



* distance b/w P_1 and P_2 = distance b/w Q and P_2 *

Choose point Q on P_1: $Q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Choose point R on P_2: $R = \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix}$



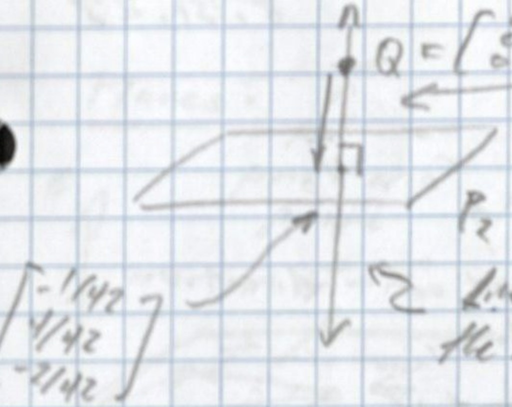
$\vec{w} = \vec{RQ} = Q - R = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$

$\text{proj}_{\vec{n}} \vec{w}$

$d = \|\text{proj}_{\vec{n}} \vec{w}\| = |\text{comp}_{\vec{n}} \vec{w}| = \left| \frac{\vec{n} \cdot \vec{w}}{\|\vec{n}\|} \right| = \left| \frac{\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}}{\sqrt{1+16+4}} \right|$

$= \frac{\frac{1}{2}}{\sqrt{21}}$

or



Find where line intersects P_2

$P_2 = 2x - 8y + 4z = -1$

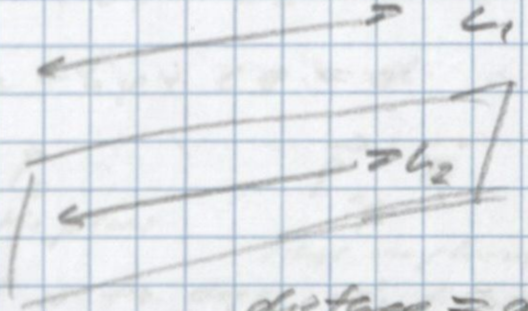
line through Q in the direction of $\vec{n}$ $\quad \ell = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} t \\ -4t \\ 2t \end{bmatrix}$

$2(t) - 8(-4t) + 4(2t) = -1 \Rightarrow 42t = -1 \Rightarrow t = -1/42$

d = distance b/w $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1/42 \\ 4/42 \\ -2/42 \end{bmatrix} = \sqrt{\left(-\frac{1}{42}\right)^2 + \left(\frac{4}{42}\right)^2 + \left(\frac{-2}{42}\right)^2}$

$\Rightarrow \dfrac{1}{2\sqrt{21}}$

Ex. Let ℓ_1 and ℓ_2 be skew lines. How do we find the minimum distance b/w a point on ℓ_1 and a point on ℓ_2?



P3 parallel to ℓ_1 and contains ℓ_2

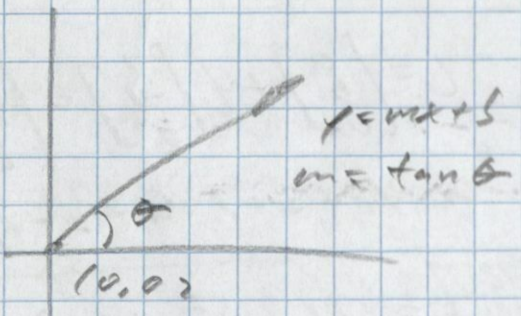
distance = distance from any point on ℓ_1 to P

Mini-lecture

8.31.23

Line:

in $\mathbb{R}^2$:



in $\mathbb{R}^3$: $\vec{r}_0$, $\vec{v}_0$

point on line, direction

* $\vec{r} = \vec{r}_0 + t\vec{v}$

(vector form)

vector form:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

* $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_0 + ta \\ y_0 + tb \\ z_0 + tc \end{bmatrix}$$

standard form:

* $$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

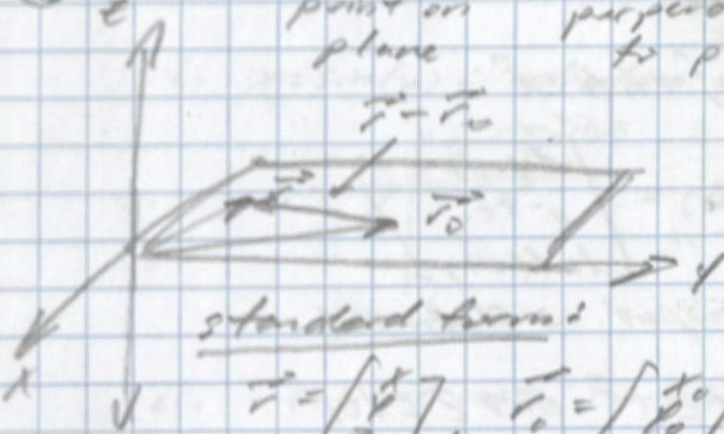
symmetric form:

$$\frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t, \quad \frac{z - z_0}{c} = t$$

* $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes: $\vec{r}_0$, $\vec{n}$

point on plane   perpendicular to plane



* $\vec{r}-\vec{r}_0 \cdot \vec{n} = 0$

(vector equation)

standard form:

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad n = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\Rightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

* $\Rightarrow ax + by + cz = d$ $d = \vec{r}_0 \cdot \vec{n}$

Distance: P_0 , $\vec{P}$ $\vec{Q}$

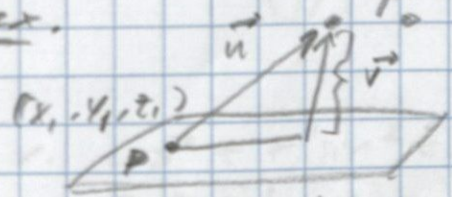
on the plane   lines on plane

ex. what is the eqn of the plane?

$\vec{n} = \vec{P} \times \vec{Q}$

$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$

ex. P_0 (x_0, y_0, z_0)



ax+by+cz=d

* $\text{dist} = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$

$$\vec{u} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

* $\text{dist} = \text{comp}_{\vec{v}} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$

[Lecture Notes] Vector Valued Functions and Space Curves 9.1.23

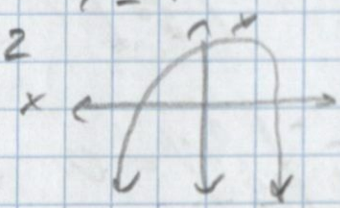
$\vec{l}(t) = \vec{P_0} + t\vec{v}$ *curve special case*

* $\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

<u>ex.</u> what is the domain of $\vec{r}(t) = \begin{bmatrix} \sqrt{4-t^2} \\ e^{-3t} \\ \ln(t+1) \end{bmatrix}$

<u>domain</u> = set of all input values such that output is defined

$\sqrt{4-t^2}$: $4-t^2 \geq 0 \iff 4 \geq t^2 \iff 2 \geq |t|$
$\Rightarrow -2 \leq t \leq 2$



e^{-3t} : defined for all t in $(-\infty, \infty)$

$\ln(t+1)$: $t+1 > 0 \iff t > -1$

domain: $-1 < t \leq 2$ or $t \in (-1, 2]$

<u>Continuity:</u>

* $\lim\limits_{t \to a} \vec{r}(t) = \langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \rangle$

* $\lim\limits_{t \to a} \vec{r}(t) = \vec{r}(a) \Rightarrow$ continuous

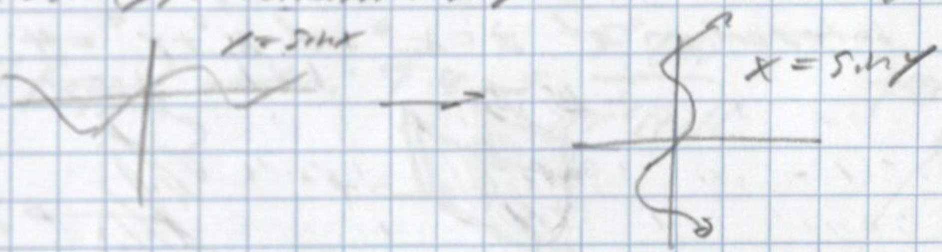
$\overset{x}{\sim\!\sim}\ \overset{y}{\sim\!\sim}$

ex. i. $\vec{r}(t) = \langle \sin t, t \rangle$ ii. $\vec{r}(t) = \langle \sqrt{2}\cos t, \sin t \rangle$ in $\mathbb{R}^2$

iii. $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ iv. $\vec{r}(t) = \langle t, \sin t, 2\cos t \rangle$ in $\mathbb{R}^3$

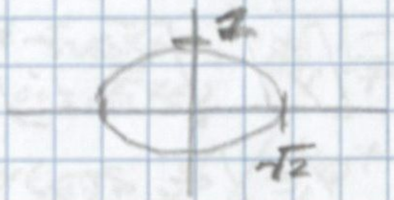
v. $\vec{r}(t) = \langle t\cos t, t\sin t, t \rangle$

i. $x = \sin(y)$: reflection of $y = \sin x$ in line $y = x$



ii. $\vec{r}(t) = \langle \cos t, \sin t \rangle \to$ unit circle

$\vec{r}(t) = \langle \sqrt{2}\cos t, \sin t \rangle \to$ stretched unit circle

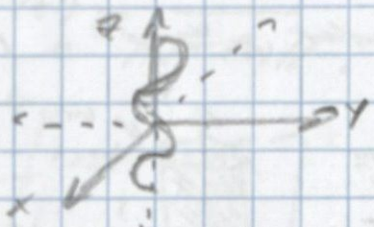


$\cos^2 t + \sin^2 t = 1$

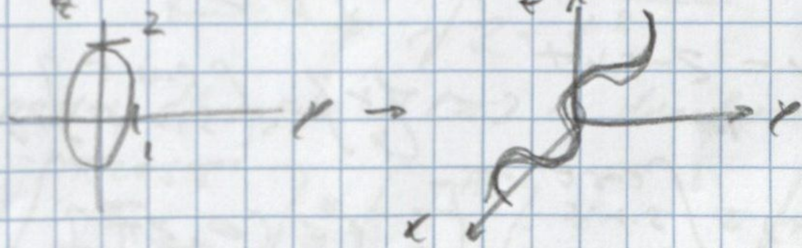
or $\Rightarrow \left(\frac{x}{\sqrt{2}}\right)^2 + y^2 = 1$ ∴ ellipse

iii. projection onto xy-plane is unit circle

z-axis makes it a circular helix on z-axis



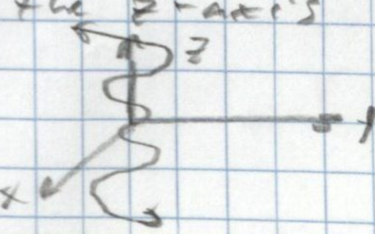
iv. elliptic helix centered in x-axis



v. $\cos^2 t + \sin^2 t = 1 \Rightarrow t^2\cos^2 t + t^2 \sin^2 t = t^2$

$\Rightarrow x^2 + y^2 = z^2$

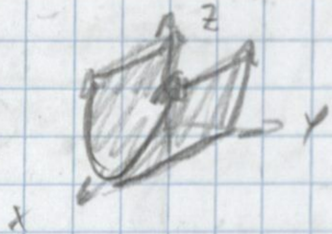
→ conical helix centered on the z-axis

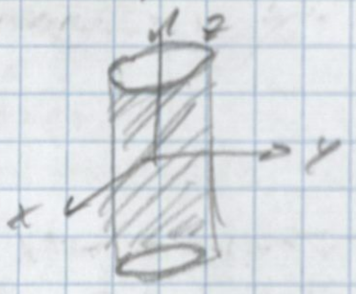


Intersection of Surfaces:

* 2 planes intersect to a line
  2 surfaces intersect to a curve

ex. find a vector valued function representing the intersection of $z = x^2$ and $x^2 + y^2 = 4$





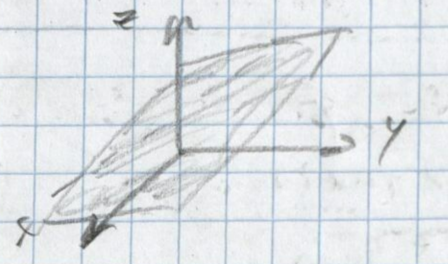
$z = x^2 = (2\cos t)^2 = 4\cos^2 t$

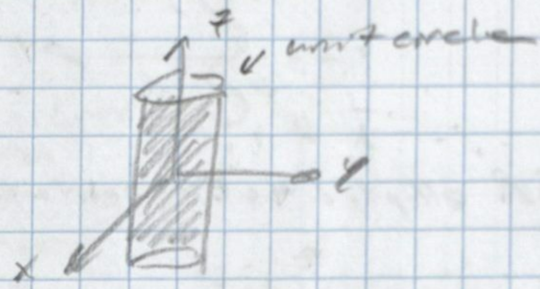
circle of $r = 2$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 4\cos^2 t \end{bmatrix}, \quad t \in [0, 2\pi)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}, \quad t \in [0, 2\pi)$$

ex. determine the parametric equations and the vector equation for the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$





$z = 2 - y = 2 - \sin t$

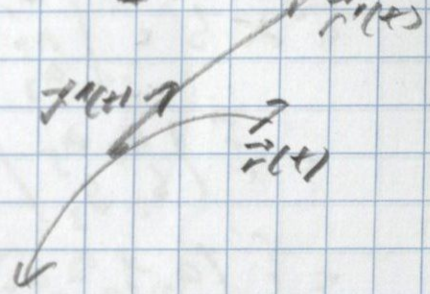
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad t \in (0, 2\pi]$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ 2 - \sin t \end{bmatrix}, \quad t \in (0, 2\pi]$$

Derivatives and Integrals:

* $\frac{d\vec r}{dt} = \vec r\,'(t) = \lim_{h \to 0} \frac{\vec r(t+h) - \vec r(t)}{h} = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$

* unit tangent vector: $\vec T(t) = \frac{\vec r\,'(t)}{\|\vec r\,'(t)\|}$



ex. Let $\vec r(t) = \begin{bmatrix} t\cos t \\ t \\ t\sin t \end{bmatrix}$

← conical helix centered on y-axis

Find i. $\vec r\,'(t)$ iii. equation for line tangent to $\vec r(t)$ at $t = \pi$
ii. $\vec T(t)$

i. $\vec r\,'(t) = \frac{d}{dt}\begin{bmatrix} t\cos t \\ t \\ t\sin t \end{bmatrix} = \begin{bmatrix} 1\cdot\cos t + t(-\sin t) \\ 1 \\ 1\cdot\sin t + t(\cos t) \end{bmatrix} = \begin{bmatrix} \cos t - t\sin t \\ 1 \\ \sin t + t\cos t \end{bmatrix}$

ii. $\vec T(t) = \frac{\vec r\,'(t)}{\|\vec r\,'(t)\|}$. $\|\vec r\,'(t)\| = \sqrt{(\cos t - t\sin t)^2 + 1^2 + (\sin t + t\cos t)^2}$

$= \sqrt{(\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t) + 1 + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t)}$

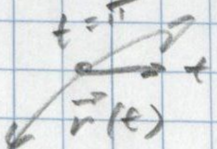
$= \sqrt{2 + t^2}$

$\therefore \vec T(t) = \frac{1}{\sqrt{2+t^2}}\begin{bmatrix} \cos t - t\sin t \\ 1 \\ \sin t + t\cos t \end{bmatrix}$

iii. line passes through $\vec r(\pi)$ in direction $\vec r\,'(\pi)$

$= \begin{bmatrix} \pi\cos\pi \\ \pi \\ \pi\sin\pi \end{bmatrix} = \begin{bmatrix} -\pi \\ \pi \\ 0 \end{bmatrix}$. $= \begin{bmatrix} -1 \\ 1 \\ -\pi \end{bmatrix}$

point direction vector



$\vec\ell(t) = \begin{bmatrix} -\pi \\ \pi \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ \pi \end{bmatrix}$

$\boxed{\text{Independent Notes}}$ Cross Product Shortcut 9.1.23

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \hat{i}\,(a_2 b_3 - a_3 b_2) - \hat{j}\,(a_1 b_3 - a_3 b_1) + \hat{k}\,(a_1 b_2 - a_2 b_1)$$

$$= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

2. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

3. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Lecture Notes

9.4.23

ex. Find the tangent vector and the unit tangent vector of the space curve

$$\vec{r}(t) = \begin{bmatrix} \cos t \\ 3t \\ 2\sin 2t \end{bmatrix} \text{ for } t = 0$$

tangent vector: $\vec{r}\,'(0)$

$$\vec{r}\,'(t) = \begin{bmatrix} -\sin t \\ 3 \\ 4\cos 2t \end{bmatrix}, \quad \vec{r}\,'(0) = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

unit tangent vector: $T = \frac{1}{\|\vec{r}\,'(0)\|}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$

properties of Derivatives:

1. $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}\,'(t)$
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] \; \vec{u}\,'(t) \cdot \vec{v}(t) +$

$\frac{d}{dt}\|\vec{r}(t)\| =$

In particular, if $\|\vec{r}(t)\|$ is constant, then $\frac{d}{dt}\|\vec{r}(t)\| = 0$, so $\vec{r}(t)$ and $\vec{r}\,'(t)$ are perpendicular. (Interpretation: if you're traveling on a sphere, then the direction is perpendicular to position).

Integration:

* $\int_a^b \vec{r}(t)\,d = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i)\Delta t_i$

$= \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$

* $\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\hat{i} + \left(\int g(t)\,dt\right)\hat{j} + \left(\int h(t)\,dt\right)\hat{k}$

ex. find $\int \vec{r}(t)\,dt$ and $\int_0^2 \vec{r}(t)\,dt$ where $\vec{r}(t) = t\hat{i} - t^3\hat{k} = \begin{bmatrix} t \\ 0 \\ -t^3 \end{bmatrix}$

$$\int \vec{r}(t)\,dt = \int \begin{bmatrix} t \\ 0 \\ -t^3 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}t^2 \\ 0 \\ -\frac{1}{4}t^4 \end{bmatrix} + \vec{C}$$

* $\left(\vec{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \text{vector of constants} \right)$

$$\int_0^2 \vec{r}(t)\,dt \underset{\substack{\uparrow \\ \text{fundamental theorem of calculus}}}{=} \begin{bmatrix} \frac{1}{2}2^2 \\ 0 \\ -\frac{1}{4}\cdot 2^4 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}0^2 \\ 0 \\ -\frac{1}{4}0^4 \end{bmatrix} = \left. \begin{bmatrix} \frac{1}{2}t^2 \\ 0 \\ -\frac{1}{4}t^4 \end{bmatrix} \right|_{t=0}^{t=2}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

Application:

angle between tangent lines = angle between tangent vectors

ex: $\vec{r}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \\ t \end{bmatrix}$, $\vec{r}_2(s) = \begin{bmatrix} -s \\ s^2-1 \\ \ln s \end{bmatrix}$.

$t=\pi$ $\qquad$ $s=1$

what is the angle at intersection at $P=(-1,0,\pi)$?

Want angle θ b/w $\vec{r}_1{}'(\pi)$ and $\vec{r}_2{}'(1)$.

$$\vec{r}_1{}'(t) = \begin{bmatrix} -\sin t \\ -\cos t \\ 1 \end{bmatrix}, \quad \vec{r}_1{}'(\pi) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \longleftarrow \theta$$

$$\vec{r}_2{}'(s) = \begin{bmatrix} -1 \\ 2s \\ \frac{1}{s} \end{bmatrix}, \quad \vec{r}_2{}'(1) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{r}_1{}'(\pi) \cdot \vec{r}_2{}'(s) = \|\vec{r}_1{}'(\pi)\| \|\vec{r}_2{}'(s)\| \cos\theta$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \sqrt{0^2+1^2+1^2}\sqrt{(-1)^2+2^2+1^2}\cos\theta$$

$$3 = \sqrt{2}\sqrt{6}\cos\theta$$

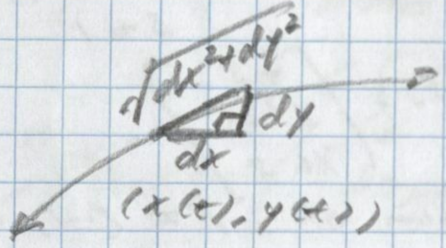
$$\cos\theta = \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\theta = 30^\circ = \frac{\pi}{6}$$

Arc Length:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

in $\mathbb{R}^2$:

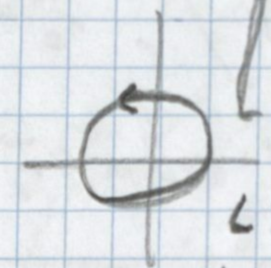


$$= \int_a^b \sqrt{dx^2 + dy^2}$$

In $\mathbb{R}^3$:

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt = \int \sqrt{dx^2 + dy^2 + dz^2}$$

$$L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$



$\begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix}$, $0 \le t \le 6\pi$

$L = 3 \cdot$ circumference

* keep track of crossing *

ex:

find the arc length of

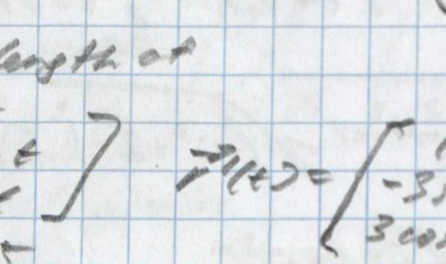
$\vec{r}(t) = \begin{bmatrix} t \\ 3\cos t \\ 3\sin t \end{bmatrix}$ $\quad \vec{r}\,'(t) = \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix}$

where $-5 \le t \le 5$

$$\|\vec{r}\,'(t)\| = \sqrt{1^2 + (-3\sin t)^2 + (3\cos^2 t)}$$

$$= \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$$

$$\int_{-5}^{5} \|\vec{r}\,'(t)\|\, dt = \int_{-5}^{5} \sqrt{10}\, dt = \sqrt{10}\, t \Big|_{t=-5}^{t=5} = 10\sqrt{10}$$



Arc Length Reparametrization (unit speed parametrization)

$$\vec{r} = \vec{r}(t(s)),\quad \underset{t=a}{\underset{\uparrow}{0}} \le s \le \underset{t=b}{\underset{\uparrow}{L}}$$

$$s(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$$

Lecture Notes

9.6.23

ex. A particle has position $\vec{r}(t) = \begin{bmatrix} t - \frac{2}{3}t^3 \\ t^2 \\ 2t \end{bmatrix}$. Find an integral to evaluate the dist traveled as it moves from $(0, 0, 0)$ to $(\frac{1}{3}, 1, 2)$.

$$\underset{t=0}{(0,0,0)} \quad \text{to} \quad \underset{t=1}{(\tfrac{1}{3}, 1, 2)}$$

want arc length from $t=0$ to $t=1$

$$\int_0^1 \|\vec{r}\,'(t)\|\,dt$$

$$\boxed{= \int_0^1 \sqrt{5+4t^4}\,dt}$$

$$\vec{r}\,'(t) = \begin{bmatrix} 1-2t \\ 2t \\ 2 \end{bmatrix}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(1-2t^2)^2 + (2t)^2 + (2)^2}$$

$$= \sqrt{(1-4t^2+4t^4)+4t^2+4} = \sqrt{5+4t^4}$$

Arc Length Reparametrization:

$$\vec{r}(t) = \vec{r}(t(s)), \quad \underset{t=a}{0} \le s \le \underset{t=b}{L}$$

L = arc length of $\vec{r}$ from $t=a$ to $t=b$

ex. Reparametrize $\vec{r}(t) = \begin{bmatrix} t \\ 3\cos t \\ 3\sin t \end{bmatrix}$, $\underset{a}{-5} \le t \le \underset{b}{5}$ w/ respect to arc length

Arc Length Parameter $s = \int_{-5}^{t} \|\vec{r}\,'(u)\|\,du = \int_{-5}^{t} \sqrt{10}\,du = \sqrt{10}\,u \Big|_{u=-5}^{u=t}$

$$= \sqrt{10}\,t - \sqrt{10}(-5)$$

$$= \sqrt{10}(t+5)$$

recall: $\vec{r}\,'(t) = \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix}$

$$\|\vec{r}\,'(t)\| = \sqrt{10}$$

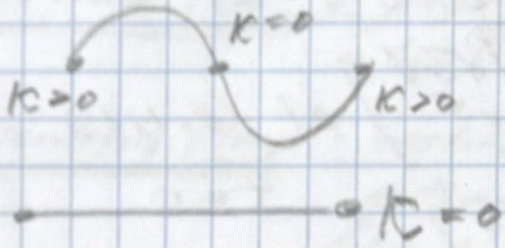
Solve for t in terms of s:

$$s = \sqrt{10}(t+5) \Rightarrow \frac{s}{\sqrt{10}} = t+5 \Rightarrow \frac{s}{\sqrt{10}} - 5 = t$$

Arc-Length (or unit speed) parametrization:

$$\vec{r}(s) = \begin{bmatrix} \frac{s}{\sqrt{10}} - 5 \\ 3\cos\left(\frac{s}{\sqrt{10}} - 5\right) \\ 3\sin\left(\frac{s}{\sqrt{10}} - 5\right) \end{bmatrix}, \quad 0 \le s \le \sqrt{10}\cdot 10$$

Curvature: The measure of how curvy something is.



intuitively,
κ = how fast direction is changing when we travel in unit speed

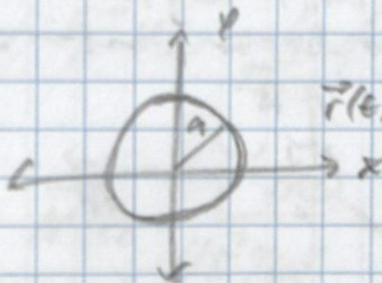
* $\kappa = \left\|\frac{d\vec{T}}{ds}\right\|$ $\vec{T}$ = unit tangent vector

magnitude of the RoC of the unit tangent vector in respect to arc length

* $\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

proof: sec 13.3, Theorem 10

ex. Find the curvature of a circle of radius a



$\vec{r}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\cos t \\ a\sin t \\ 0 \end{bmatrix}, \quad 0 \le t \le 2\pi$

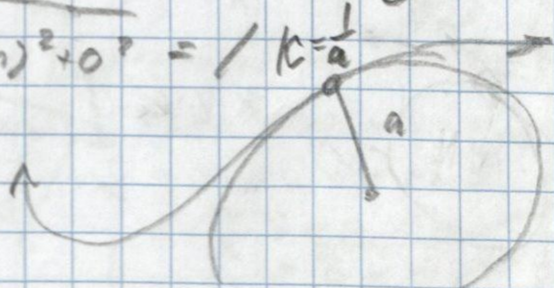
$\vec{r}'(t) = \begin{bmatrix} -a\sin t \\ a\cos t \\ 0 \end{bmatrix}, \quad \|\vec{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + 0^2} = \sqrt{a^2} = a$

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{a}\begin{bmatrix} -a\sin t \\ a\cos t \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}, \quad \vec{T}'(t) = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}$

$\|\vec{T}'(t)\| = \sqrt{(-\cos t)^2 + (-\sin)^2 + 0^2} = 1$

$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{1}{a}$

$\kappa = \frac{1}{a}$



"circle of best fit"

more effort than needed, so:

$$* \ \kappa = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

Ex. Find the curvature of $\vec{r}(t) = \begin{bmatrix} \sqrt{2}\,t \\ e^{t} \\ e^{-t} \end{bmatrix}$ at $\underbrace{(0,1,1)}_{t=0}$

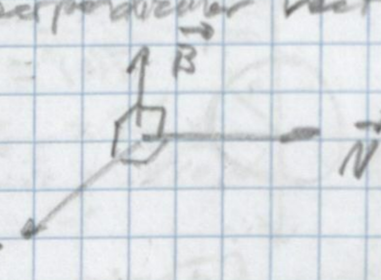
$$\vec{r}\,'(t) = \begin{bmatrix} \sqrt{2} \\ e^{t} \\ -e^{-t} \end{bmatrix}, \quad \vec{r}\,'(0) = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{r}\,''(t) = \begin{bmatrix} 0 \\ e^{t} \\ e^{-t} \end{bmatrix}, \quad \vec{r}\,''(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\kappa = \frac{\left\| \begin{bmatrix} 2 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} \sqrt{2} \\ 1 \\ -1 \end{bmatrix} \right\|^3} = \frac{\sqrt{2^2 + (-\sqrt{2})^2 + (\sqrt{2})^2}}{\sqrt{(\sqrt{2})^2 + 1^2 + (-1)^2}^{\,3}} = \frac{\sqrt{8}}{\sqrt{4}^{\,3}} = \frac{\sqrt{8}}{8} = \frac{1}{\sqrt{8}}$$

__Frenet - Serret Frame__ "T-N-B __Frame__" 3 unit-length perpendicular vectors



__Unit Normal Vector:__

$$* \ \vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

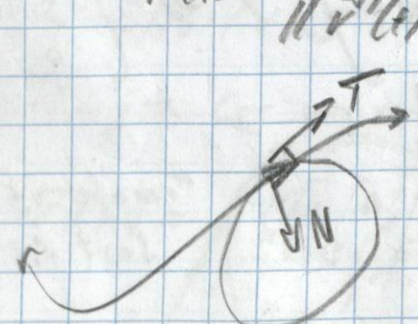
$\vec{T} \cdot \vec{N} = 0$ since $\|\vec{T}\|$ is constant

__Binormal Vector:__

$$* \ \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\vec{B}(t) = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}, \quad \vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$$

$$\vec{T}(t) = \vec{N}(t) \times \vec{B}(t)$$



* like $\hat{i}, \hat{j}$, and $\hat{k}$ but rotatable *

T-N-B Planes:

Normal Plane: perpendicular to $\vec{r}'(t)$ and orthogonal to $\vec{T}$

Osculatory Plane: best captures the motion of a curve, orthogonal to $\vec{B}$
(limit of 3 points on the curve as the points collide)

Rectifying Plane: orthogonal to $\vec{N}$, won't be used much

ex: Find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$ for $\vec{r}(t) = \begin{bmatrix} t \\ 3\cos t \\ 3\sin t \end{bmatrix}$.
find equations for normal and osculatory planes at $(\frac{\pi}{2}, 0, 3)$ → $(\perp T \text{ or } \perp \vec{r}')$ → $(\perp B)$

Recall: $\vec{r}'(t) = \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix}$, $\|\vec{r}'(t)\| = \sqrt{10}$,

$$\vec{T}(t) \boxed{= \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix}}$$

$$\vec{T}'(t) = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ -3\cos t \\ -3\sin t \end{bmatrix}, \quad \|\vec{T}'(t)\| = \frac{1}{\sqrt{10}} \sqrt{0^2 + (-3\cos t)^2 + (-3\sin t)^2} = \frac{3}{\sqrt{10}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ -3\cos t \\ -3\sin t \end{bmatrix}}{\frac{3}{\sqrt{10}}} \boxed{= \begin{bmatrix} 0 \\ -\cos t \\ -\sin t \end{bmatrix}}$$

$$\vec{B}(t) = \vec{T} \times \vec{N} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix} \times \begin{bmatrix} 0 \\ -\cos t \\ -\sin t \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3\sin^2 t + 3\cos^2 t \\ \sin t \\ -\cos t \end{bmatrix} \boxed{= \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ \sin t \\ -\cos t \end{bmatrix}}$$

normal: $\perp \vec{T} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3\sin t \\ 3\cos t \end{bmatrix} \propto \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

"$\propto$ is proportional to"

→ $x - 3y = d$

to find d, plug in $(\frac{\pi}{2}, 0, 3)$

$\frac{\pi}{2} - 3(0) = d$ → $\boxed{x - 3y = \frac{\pi}{2}}$

osculatory plane: $\perp \vec{B} \propto \begin{bmatrix} 3 \\ \sin t \\ -\cos t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

→ $3x + y = d$

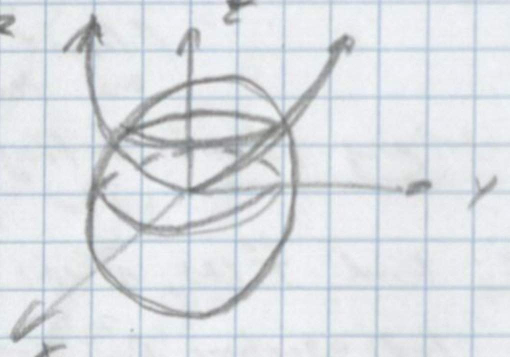
$3(\frac{\pi}{2}) + 0 = d$

→ $\boxed{3x + y = \frac{\pi}{2}}$

Tutorial Notes 9.7.23

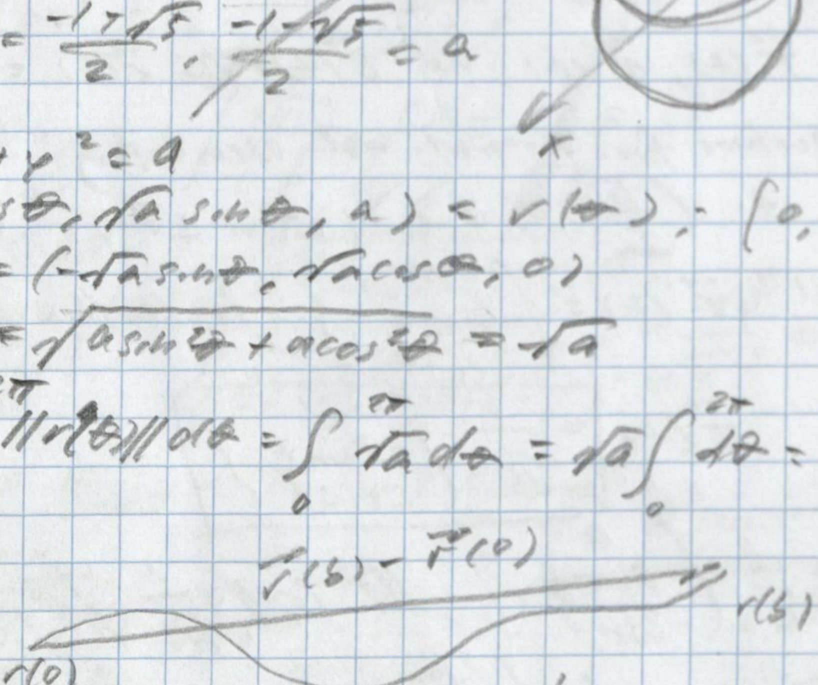
ex. parametrize the curve that intersects $x^2+y^2+z^2=1$ and $x^2+y^2=z$, then bound so that the curve is not being traced more than once. find the length of the curve.

$$x^2+y^2=1-z^2 \longrightarrow 1$$
$$x^2+y^2=z \longrightarrow 2$$

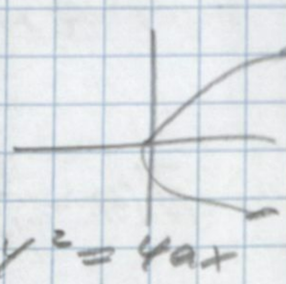


$$1-z^2=z$$
$$z^2+z-1=0$$
$$\Rightarrow z=\frac{-1+\sqrt5}{2},\ \cancel{\frac{-1-\sqrt5}{2}}=a$$

$$x^2+y^2=a$$
$$(\sqrt a\cos\theta,\ \sqrt a\sin\theta,\ a)=r(\theta),\ [0,2\pi]$$
$$r'(\theta)=(-\sqrt a\sin\theta,\ \sqrt a\cos\theta,\ 0)$$
$$\|r'(\theta)\|=\sqrt{a\sin^2\theta+a\cos^2\theta}=\sqrt a$$
$$l=\int_0^{2\pi}\|r'(\theta)\|\,d\theta=\int_0^{2\pi}\sqrt a\,d\theta=\sqrt a\int_0^{2\pi}d\theta=\sqrt a\,\theta\Big|_0^{2\pi}=2\pi\sqrt a$$

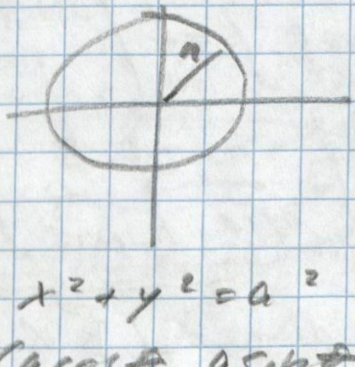


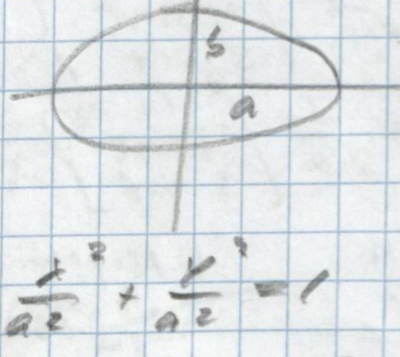
$$\int_0^b\|r'(t)\|dt$$

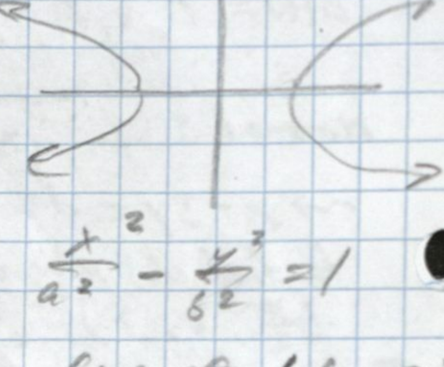


$$y^2=4ax$$
$$y=t,\ x=4at$$

Parametrization:







$x^2+y^2=a^2$

$(a\cos\theta,\ a\sin\theta)$

$\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$

$(a\cos\theta,\ b\sin\theta)$

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

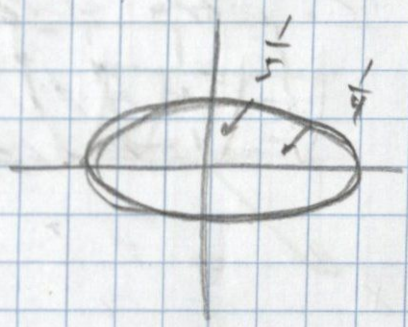
$(a\sec\theta,\ b\tan\theta)$

ex: $16x^2 + 25y^2 = 1$, $z = e^{x}$

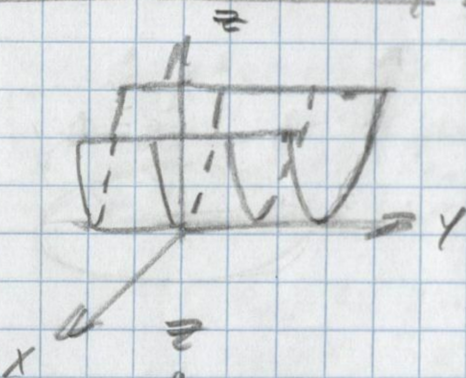
$$\frac{x^2}{\frac{1}{16}} + \frac{y^2}{\frac{1}{25}} = 1$$

$x = \frac{1}{4}\cos\theta$ $\quad z = e^{\frac{1}{4}\cos\theta}$

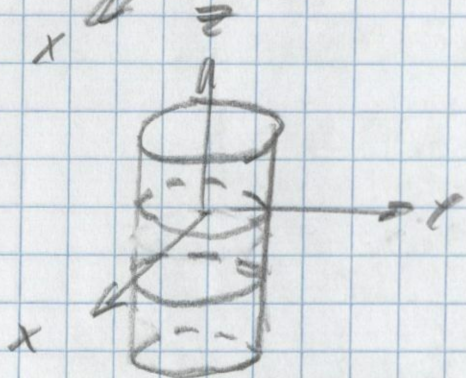
$y = \frac{1}{5}\sin\theta$



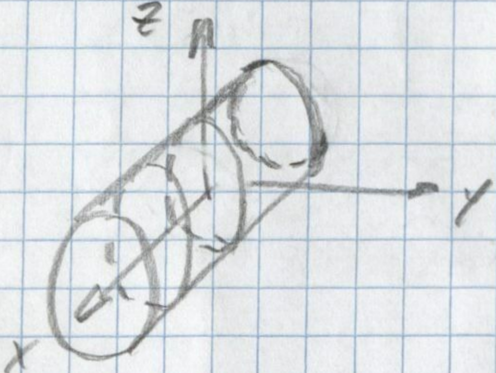
[Independent Notes] Cylinders and Quadric Surfaces 9.7.23



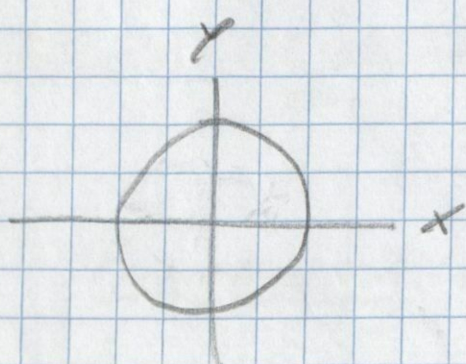
surface $z = x^2$
— parabolic cylinder



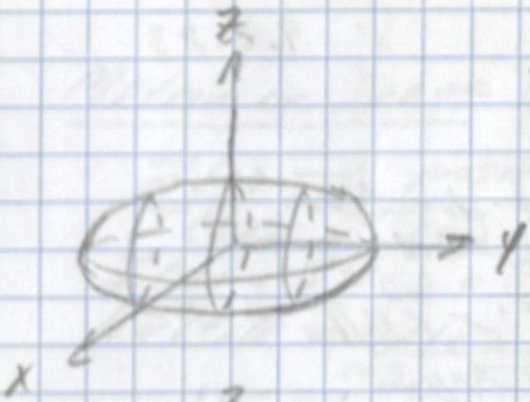
surface $x^2+y^2=1$
— circular cylinder



surface $y^2+x^2=1$
— circular cylinder

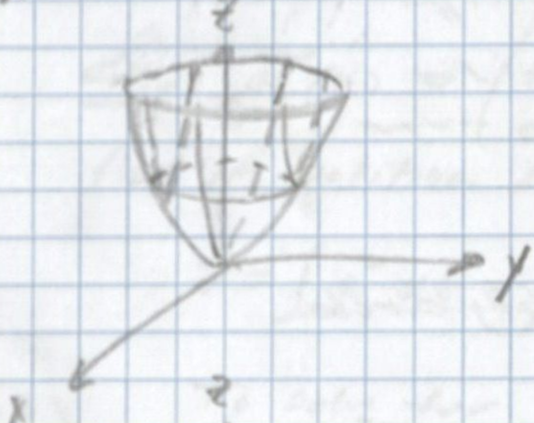


equation $y^2+x^2=1$
— only a circle in $\mathbb{R}^2$



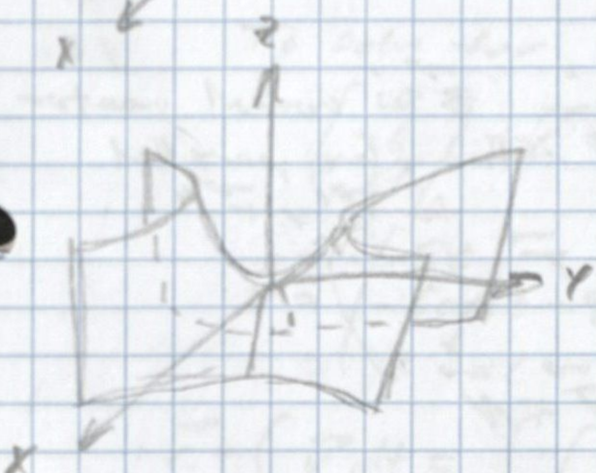
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- ellipsoid
- if $a = b = c$ then sphere



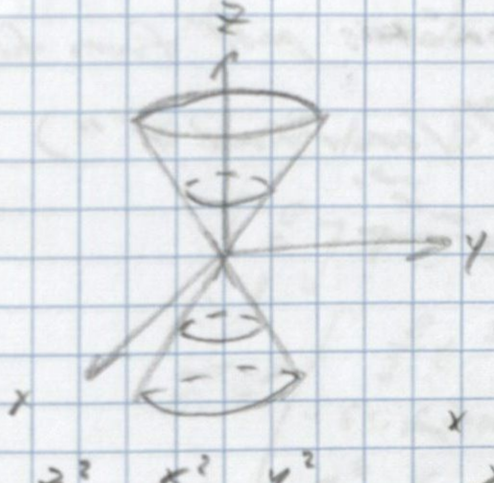
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- elliptic paraboloid



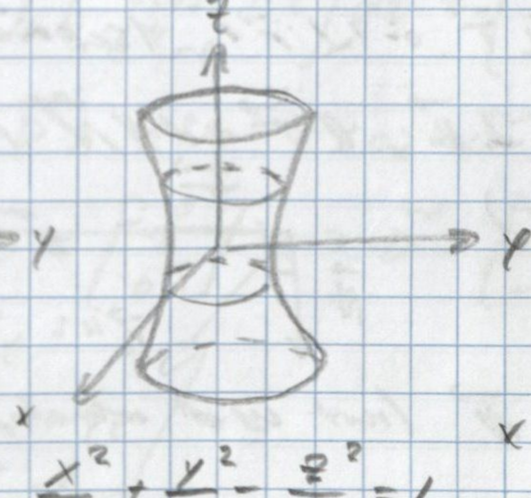
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

- hyperbolic paraboloid



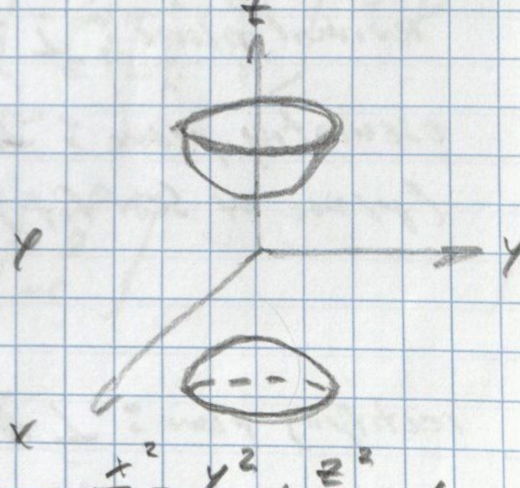
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- cone



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- hyperboloid of one sheet



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- hyperboloid of two sheets

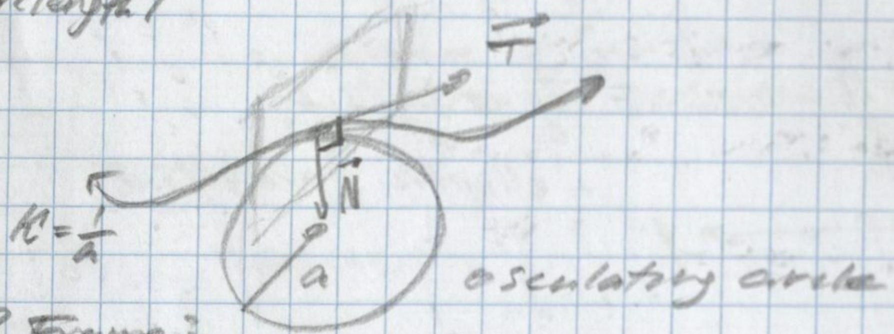
$\boxed{\text{Lecture Notes}}$ 9.8.23

Curvature:

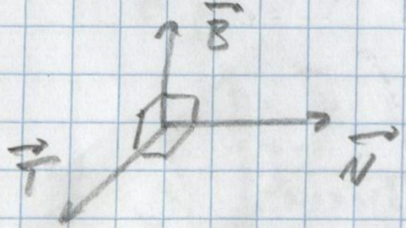
κ = magnitude of the ROC of direction when traveling at unit speed

$$= \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{T}'\|}{\|\vec{r}'\|} = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$$

arclength?



T-N-B Frame:



$$\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|}$$

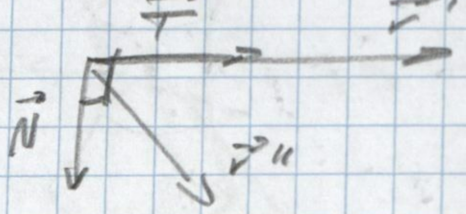
$\vec{B}$ is normal vector for plane parallel to $\vec{r}', \vec{r}''$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \vec{B} \times \vec{T}$$

$$\vec{B} = \frac{\vec{r}' \times \vec{r}''}{\|\vec{r}' \times \vec{r}''\|} = \vec{T} \times \vec{N}$$

normal plane: $\perp \vec{T}$ or $\perp \vec{r}'$ (separates past from future)

osculating plane: $\perp \vec{B}$, $\parallel \vec{T}$ and $\parallel \vec{N}$ (or $\vec{r}'$ and $\vec{r}''$)
(plane of best fit)



rectifying plane: $\perp \vec{N}$ (not used much)

<u>Motion in Space</u>

$\vec{r}(t) \rightarrow$ position

$\vec{v}(t) = \vec{r}\,'(t) \rightarrow$ velocity

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) \rightarrow$ acceleration

$\|\vec{v}(t)\| = \nu(t) \rightarrow$ speed

"nu"

<u>ex.</u> $\vec{a}(t) = \begin{bmatrix} 4t \\ 6\sin t \\ e^t \end{bmatrix}$, $\vec{v}(0) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $\vec{r}(0) = \vec{0}$,

find its position function:

$$\vec{v} = \int \vec{a}\, dt = \int \begin{bmatrix} 4t \\ 6\sin t \\ e^t \end{bmatrix} dt = \begin{bmatrix} 2t^2 \\ -6\cos t \\ e^t \end{bmatrix} + \vec{C}$$

To solve for $\vec{C}$, use $\vec{v}(0) = \langle 0, 3, 0 \rangle$:

$$\begin{bmatrix} 0 \\ -6 \\ 1 \end{bmatrix} + \vec{C} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \vec{C} = \begin{bmatrix} 0 \\ 9 \\ -1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 2t^2 \\ -6\cos t + 9 \\ e^t - 1 \end{bmatrix}$$

$$\vec{r} = \int \vec{v}\, dt = \int \begin{bmatrix} 2t^2 \\ -6\cos t + 9 \\ e^t - 1 \end{bmatrix} dt = \begin{bmatrix} \frac{2}{3}t^3 \\ -6\sin t + 9t \\ e^t - t \end{bmatrix} + \vec{D}$$

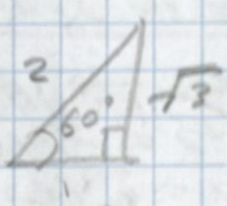
To solve for $\vec{D}$, use $\vec{r}(0) = \vec{0}$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \vec{D} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{D} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} \frac{2}{3}t^3 \\ -6\sin t + 9t \\ e^t - t - 1 \end{bmatrix}$$

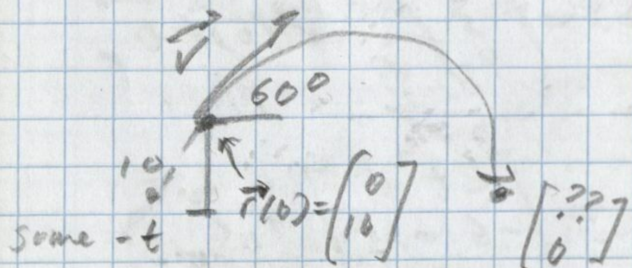
Newton's Second Law:

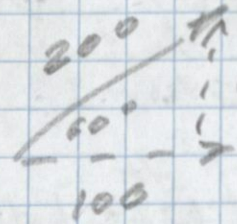
$\vec{F}(t) = m\vec{a}(t)$



$\|\vec{v}\| = 200$

ex: proj is fired at 200 m/s at $\theta = 60°$. If fired 10m above ground, what is horizontal distance covered?





$\Rightarrow \vec{v} = \begin{bmatrix} 100 \\ 100\sqrt{3} \end{bmatrix}$

$\vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}, \; g \approx 9.8$

$$\vec{v} = \int \vec{a}\,dt = \int \begin{bmatrix} 0 \\ -g \end{bmatrix} dt = \begin{bmatrix} 0 \\ -gt \end{bmatrix} + \vec{C}$$

$$\vec{v}(0) = \begin{bmatrix} 100 \\ 100\sqrt{3} \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \vec{C} = \begin{bmatrix} 100 \\ 100\sqrt{3} \end{bmatrix}$$

$$\Rightarrow \vec{C} = \begin{bmatrix} 100 \\ 100\sqrt{3} \end{bmatrix} \rightsquigarrow \vec{v} = \begin{bmatrix} 100 \\ -gt + 100\sqrt{3} \end{bmatrix}$$

$$\vec{r} = \int \vec{v}\,dt = \int \begin{bmatrix} 100 \\ -gt + 100\sqrt{3} \end{bmatrix} dt = \begin{bmatrix} 100t \\ -\frac{gt^2}{2} + 100\sqrt{3}\,t \end{bmatrix} + \vec{D}$$

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 10 \end{bmatrix} : \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \vec{D} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

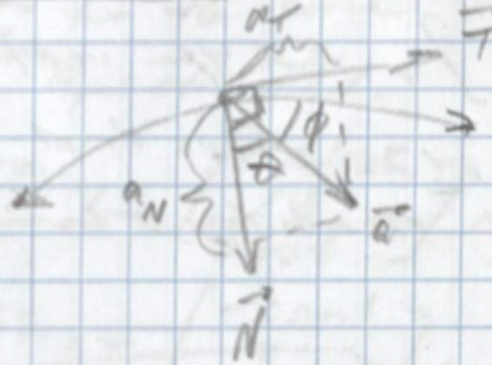
$$\Rightarrow \vec{D} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \rightsquigarrow \vec{v} = \begin{bmatrix} 100t \\ -\frac{gt^2}{2} + 100\sqrt{3}\,t + 10 \end{bmatrix}$$

↑ find when $= 0$

$t \approx 35.4$

horizontal distance $= 100t \approx 3540$

accelerations



$$\vec a = a_T \vec T + a_N \vec N$$

* $a_T = \mathrm{comp}_{\vec T}\, \vec a = \dfrac{\vec a \cdot \vec T}{\|\vec T\|} = \dfrac{\vec r\,' \cdot \vec r\,''}{\|\vec r\,'\|}$

$= \dfrac{d}{dt}\|\vec v\| = \dfrac{d}{dt} v$ ← speed

project onto tangent line

osculating plane is parallel to $\vec v$ and $\vec a$

* $a_N = \mathrm{comp}_{\vec N}\, \vec a = \dfrac{\vec a \cdot \vec N}{\|\vec N\|} = \|\vec a\|\|\vec N\|\cos\theta$

$= \|\vec a\| \sin\phi$

$= \|\vec r\,''\| \sin\phi$

$= \underbrace{\dfrac{\|\vec r\,'(t) \times \vec r\,''(t)\|}{\|\vec r\,'(t)\|}}_{\text{always } \geq 0}$

* $\|\vec a\| = \sqrt{a_T^2 + a_N^2}$ ← pythagorous
* $a_T = \dfrac{\vec r\,' \cdot \vec r\,''}{\|\vec r\,'\|}$
* $a_N = \sqrt{\|\vec a\|^2 - a_T^2}$

ex. Find a_N and a_T moving across:

$$\vec r(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}.$$

$$\vec r\,'(t) = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}, \quad \vec r\,''(t) = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}$$

$$a_T = \frac{\vec r\,' \cdot \vec r\,''}{\|\vec r\,'\|} = \frac{\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}}{\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}} = \frac{\sin t \cos t - \sin t \cos t + 0}{\sqrt{2}} = 0$$

our particle is traveling at constant speed $\left(\frac{d}{dt} v = 0\right)$

$$a_N = \sqrt{\|\vec a\|^2 - a_T^2} = \|\vec a\| = \left\| \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix} \right\|$$

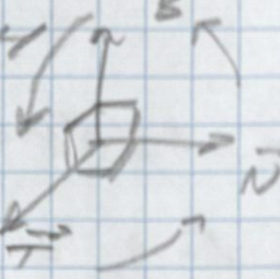
$a_N = 1$

Lecture Notes 9.11.23

ex. for the curve $\vec{r}(t) = \begin{bmatrix} t^2 \\ \frac{2}{3}t^3 \\ t \end{bmatrix}$, find the vectors $\vec{T}, \vec{N}, \vec{B}$ at the point $(1, \frac{2}{3}, 1)$. Also find equations of the normal and osculating planes at the same point.

$$\vec{r}\,'(t) = \begin{bmatrix} 2t \\ 2t^2 \\ 1 \end{bmatrix}, \quad \|\vec{r}\,'(t)\| = \sqrt{(2t)^2 + (2t^2)^2 + 1} = \sqrt{4t^4 + 4t^2 + 1}$$

$$\vec{r}\,'(1) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \|\vec{r}\,'(1)\| = \sqrt{4+4+1} = 3$$



$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Next, find $\vec{B} \propto \vec{r}\,' \times \vec{r}\,''$. $\vec{r}\,''(t) = \begin{bmatrix} 2 \\ 4t \\ 0 \end{bmatrix}$

$$\vec{B}(1) \propto \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \qquad \vec{r}\,''(1) = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{B}(1) = \frac{\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}}{\sqrt{(-4)^2 + 2^2 + 4^2}} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{N} = \vec{B} \times \vec{T} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -3 \\ 6 \\ -6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

---

or $\vec{N} = \dfrac{\vec{T}\,'}{\|\vec{T}\,'\|} \propto \vec{T}\,'$. $\vec{T}(t) = \dfrac{\begin{bmatrix} 2t \\ 2t^2 \\ 1 \end{bmatrix}}{2t^2+1} = \begin{bmatrix} 2t/2t^2+1 \\ 2t^2/2t^2+1 \\ 1/2t^2+1 \end{bmatrix}$

---

normal plane: $\perp \vec{T}$ (or $\perp r'$), through $(1, \frac{2}{3}, 1)$
(normal to motion)
Osculating plane: $\perp \vec{B}$ (or $\perp \vec{r}\,' \times \vec{r}\,''$)

Ex. for the curve $\vec{r}(t) = \begin{bmatrix} 3\cos t \\ t^2+1 \\ 3\sin t \end{bmatrix}$, find a_T at $t=1$.

$$\vec{r}\,'(t) = \begin{bmatrix} -3\sin t \\ 2t \\ 3\cos t \end{bmatrix}, \quad \vec{r}\,''(t) = \begin{bmatrix} -3\cos t \\ 2 \\ -3\sin t \end{bmatrix}$$

$$\vec{r}\,'(1) = \begin{bmatrix} -3\sin 1 \\ 2 \\ 3\cos 1 \end{bmatrix}, \quad \vec{r}\,''(1) = \begin{bmatrix} -3\cos 1 \\ 2 \\ -3\sin 1 \end{bmatrix}$$

$$a_T = \frac{\vec{r}\,'(1)\cdot\vec{r}\,''(1)}{\|\vec{r}\,'(1)\|} = \frac{\begin{bmatrix} -3\sin 1 \\ 2 \\ 3\cos 1 \end{bmatrix}\cdot\begin{bmatrix} -3\cos 1 \\ 2 \\ -3\sin 1 \end{bmatrix}}{\sqrt{(-3\sin 1)^2+2^2+(3\cos 1)^2}}$$

$$= \frac{4}{\sqrt{9+4}} = \frac{4}{\sqrt{13}}$$

<u>Functions of Several Variables:</u>

* $\{f(x,y) \mid (x,y) \in D\}$
* $z = f(x,y) \rightarrow$ surface in $\mathbb{R}^3$

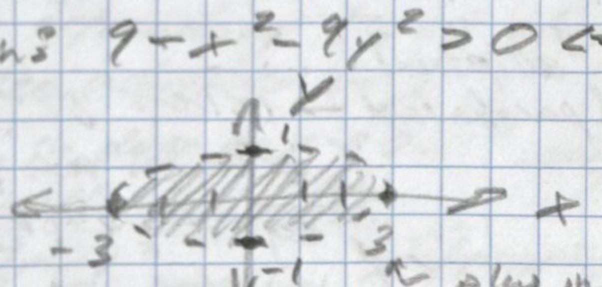
$f(x,y)$ ← independent vars

z ← dependent var

range: $(-\infty, \ln(9)]$

<u>ex.</u> Find the domain of $f(x,y) = \ln(9-x^2-9y^2)$ and sketch it

domain: $9-x^2-9y^2 > 0 \iff x^2+9y^2 < 9$

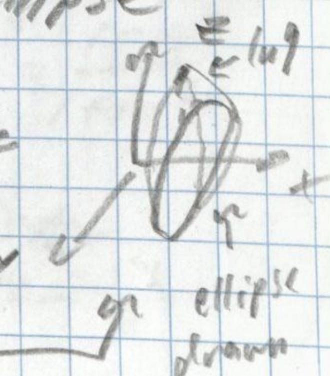


slice parallel to xy for fixed z, get an ellipse

domain is the strict interior of ellipse $x^2+9y^2=9$

Sketch surface: $z = \ln(9-x^2-9y^2)$

$\iff e^z = 9-x^2-9y^2 \iff x^2-9y^2 = 9-e^z$ ⟶



Lecture Notes

9.13.23

ex. For the curve $\vec{r}(t) = \begin{pmatrix} t^2 \\ \frac{2}{3}t^3 \\ t \end{pmatrix}$, find a_T and a_N at $t=1$.

$$\vec{r}\,'(t) = \begin{pmatrix} 2t \\ 2t^2 \\ 1 \end{pmatrix}, \quad \vec{r}\,''(t) = \begin{pmatrix} 2 \\ 4t \\ 0 \end{pmatrix}$$

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{\|\vec{r}\,'\|} = \frac{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}{\sqrt{2^2+2^2+1^2}} = \frac{4+8+0}{\sqrt{9}} = 4$$

$(= \text{comp}_{\vec{T}}\,\vec{a})$

$$a_N = \sqrt{\|\vec{a}\|^2 - a_T^2} = \sqrt{20-4^2} = 2$$

$(\|\vec{a}\|^2 = \|\vec{r}\,''\|^2 = 2^2+4^2+0^2 = 20)$

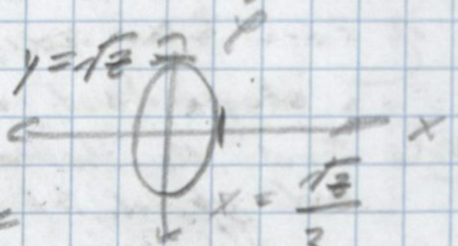
(or $a_N = \text{comp}_{\vec{N}}\,\vec{a} = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|}$)

ex. sketch the graph of $h(x,y) = 4x^2+y^2$

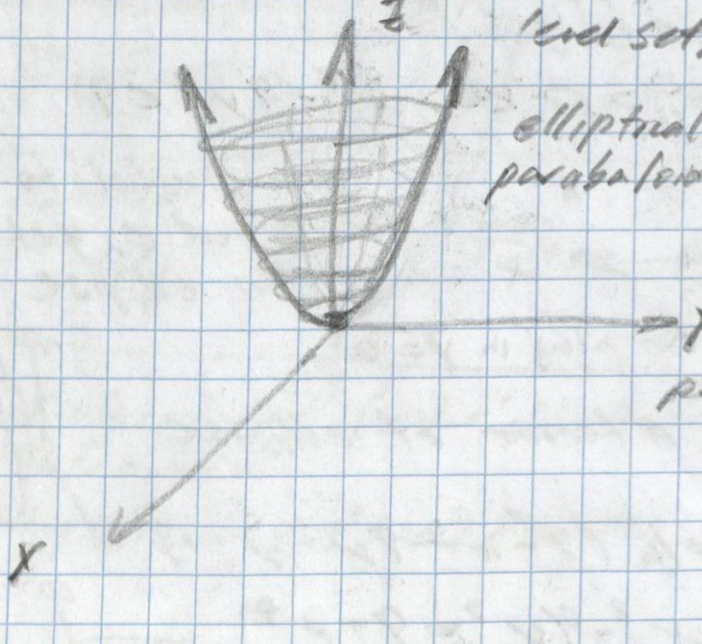
domain: $\mathbb{R}^2$ (whole plane) range $= [0,\infty)$ (all non-negative reals)

sketch: $z = 4x^2+y^2$

If we plug in a value ~~for~~ z $(z \geq 0)$



level set / slice



elliptical paraboloid → (slices parallel to z-axis are parabolas)

eg. if we plug in a y-value, we get a parabola in xz-plane.

Contour Map: plot of the level sets $4x^2+y^2=c$ for various choices of c, in xy-plane

level sets for $c>0$ are empty

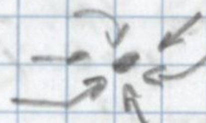
temperature / topology graphs use this

Functions of Three or More Variables:

* use level sets $\to f(x,y,z)=c$
* ex, we can think of $f(x,y,z)=c$ as a surface defined implicitly. eg. $x^2+y^2+z^2=1$ sphere

Limits and Continuity:

* $\lim_{(x,y)\to(a,b)} f(x,y)=L$

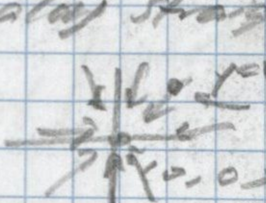


"values f are squeezed towards L"

* its easier to show limits don't exist *
  (b/c there are so many paths to L)

Ex. does $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exist?

1. Let's hope the limit does not exist.
2. Let's try approaching $(0,0)$ along straight lines $y=mx$ (for arbitrary m)
3. As $x\to 0$, so does $mx\to 0$.



So if lim exist, it equals

$$\lim_{x\to 0}\frac{x^2-(mx)^2}{x^2+(mx)^2}=\lim_{x\to 0}\frac{x^2(1-m^2)}{x^2(1+m^2}=\lim_{x\to 0}\frac{1-m^2}{1+m^2}=\frac{1-m^2}{1+m^2}$$

or use L'Hospital's rule

This lim depends on m, so limit does not exist.

If lim did not depend on m, limit would STILL not exist, due to non-linear paths to L.

* To show a limit <u>does</u> exist, use squeeze theorem:

* if $L(x,y) \le f(x,y) \le U(x,y)$, and

$$\lim_{(x,y)\to(a,b)} L(x,y) = \lim_{(x,y)\to(a,b)} U(x,y), \text{ then } \lim_{(x,y)\to(a,b)} f(x,y) = \text{same limit}$$

<u>ex.</u> does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

Lets try approaching along the line $y=mx$.
If the limit exists, then it equals

$$\lim_{x\to 0} \frac{x(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m)} = \frac{m}{1+m^2}.$$

This depends on m, so the limit does <u>not</u> exist

<u>ex.</u> does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Lets try $y=mx$

$$\lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} \underset{\text{cancel } x^2}{=} \lim_{x\to 0} \frac{xm^2}{1+m^4x^2} = \frac{0\cdot m^2}{1+m^4\cdot 0^2} = 0.$$

(using geogebra it is clear lim DNE), so:

If we let $u=y^2$, then $u\to 0+$ when $y\to 0$, so

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,u)\to(0,0^+)} \frac{xu}{x^2+u^2}$$ which doesn't exist by ex. 4.

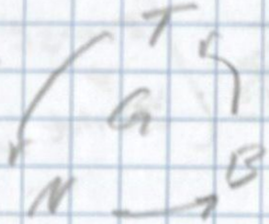
[Tutorial Notes] 9.14.23

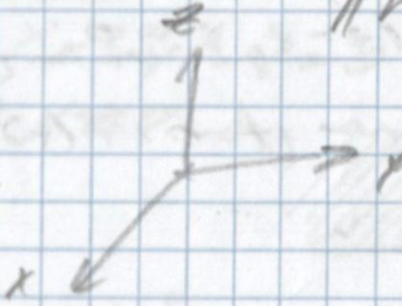
TNB:

$$\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

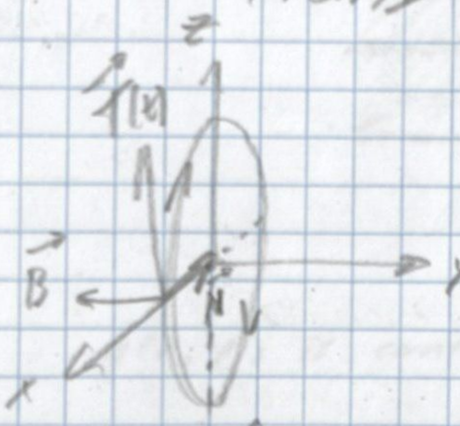
$$\vec{N} = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

$$\vec{B} = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}$$





ex: Find T, N, and B for $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 0 \\ 2\sin t \end{bmatrix}$ at $t=0$.
(verify visually)



$$\vec{r}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

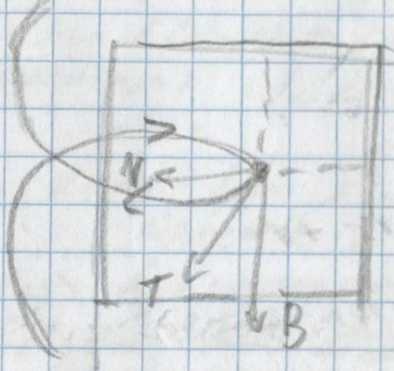
$$\vec{r}\,'(t) = \begin{bmatrix} -2\sin t \\ 0 \\ 2\cos t \end{bmatrix}, \quad \vec{r}\,'(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

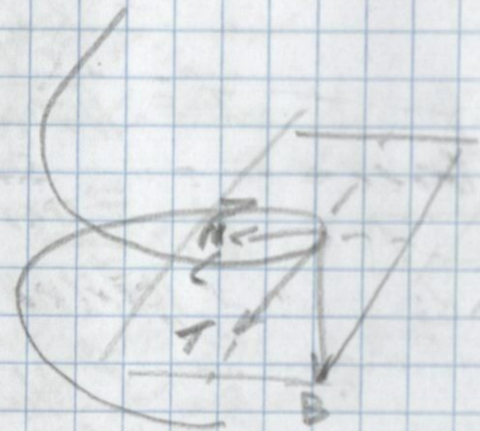
$$\vec{T}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{r}\,''(t) = \begin{bmatrix} -2\cos t \\ 0 \\ -2\sin t \end{bmatrix}, \quad \vec{r}\,''(0) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{B}(0) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{r}\,'(0) \times \vec{r}\,''(0) = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$$

$$\vec{N}(0) = \vec{B}(0) \times \vec{T}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

TNB Planes:





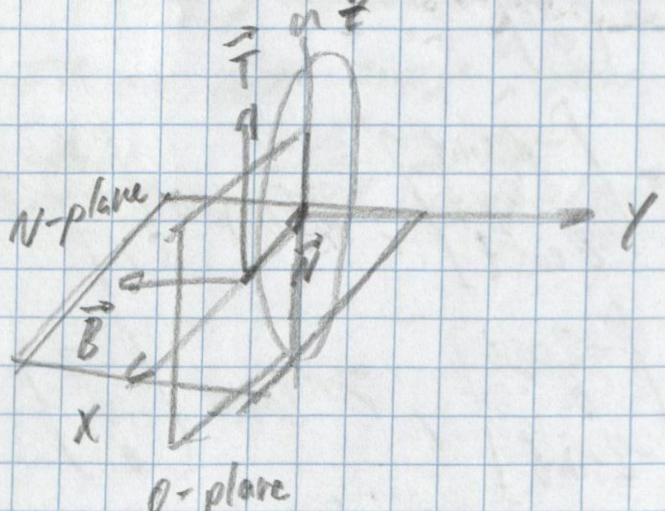
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

ex. Find N and O Plane for $\vec{r}(t)=\begin{bmatrix}2\cos t\\0\\2\sin t\end{bmatrix}$ at $t=0$.

$$\vec{r}(0)=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

N Plane: $z=0 \rightarrow$ xy-plane

O Plane: $y=0 \rightarrow$ xz-plane

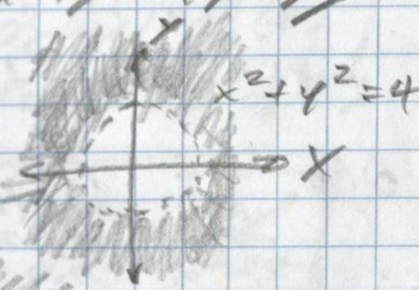


Lecture Notes 4.15.23

ex: does $\lim_{(x,y)\to(0,0)} \frac{4-xy}{x^2+3y^2+1}$ exist?

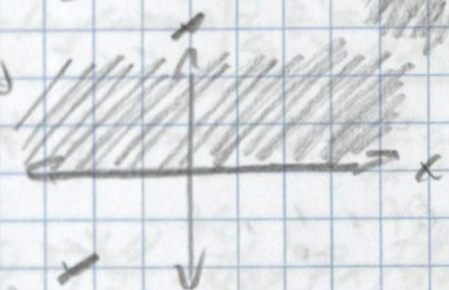
Plug in $(x,y)=(0,0)$: $\frac{4-0}{0+0+1}=4$

ex: determine whether $g(x,y)=\ln(x^2+y^2-4)-\sqrt{y}$ is continuous?

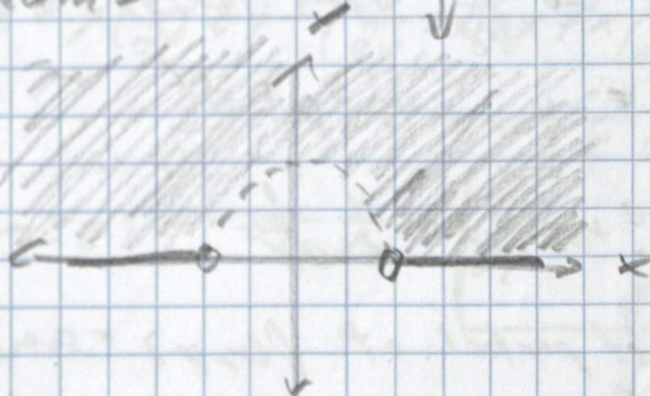


domain:

$$\begin{cases} x^2+y^2-4>0 \\ y\geq 0 \end{cases}$$



domain:



g(x,y) is continuous precisely on its domain

Partial Derivatives:

with respect to x:

$$* \quad \partial_x f = \frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

* derivative respect to x when y is constant *

with respect to y:

$$* \quad \partial_y f = \frac{\partial f}{\partial y} = f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

∂ = "del", "dy", "dee", "partial"

ex. find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x,y) = x^3 + 2x^2y + x^4y^2 + \sqrt{y}$

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy + 4x^3y^2$$

$$\frac{\partial f}{\partial y} = 2x^2 + 2x^4y + \frac{1}{2}\frac{1}{\sqrt{y}}$$

ex. find the first partial of $f(x,y) = \sin(x\cos y)$

$$\partial_x f = \frac{\partial f}{\partial x} = f_x = \cos(x\cos y)\cos y$$

$$\left(= \cos(x\cos y)\frac{\partial}{\partial x} x\cos y \text{ by chain rule}\right)$$

$$\partial_y f = \frac{\partial f}{\partial y} = f_y = \cos(x\cos y)(-x\sin y)$$

$$\left(= \cos(x\cos y)\frac{\partial}{\partial y}(x\cos y)\right)$$

ex. find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where z is implicitly defined by $yz + x\ln y = z^2$

$(z = z(x,y))$

Take partials of both sides: *product rule*

$$\partial_x : (\partial_x y)z + y(\partial_x z) + (\partial_x x)\ln y + x(\partial_x \ln y) = \partial_x (z)^2$$

$$\Rightarrow 0 + y(\partial_x z) + (1)\ln y + x(0) = 2z(\partial_x z)$$

solve for $\partial_x z$:

$$\partial_x z\,(y - 2z) = -\ln y \Rightarrow \boxed{\partial_x z = \frac{\partial z}{\partial x} = \frac{-\ln y}{y - 2z}}$$

* product rule: $\frac{\partial}{\partial x}(fg) = \left(\frac{\partial}{\partial x} f\right) g + f\left(\frac{\partial}{\partial x} g\right)$ *

$$\frac{\partial}{\partial y} : \left(\frac{\partial}{\partial y} y\right) z + y\left(\frac{\partial}{\partial y} z\right) + \left(\frac{\partial}{\partial y} x\right)\ln y + x\left(\frac{\partial}{\partial y} \ln y\right) = \frac{\partial}{\partial y} z^2$$

$$1 \cdot z + y \cdot \frac{\partial z}{\partial y} + 0 + x\left(\frac{1}{y}\right) = 2z\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y}(y - 2z) = -z - \frac{x}{y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-z - \frac{x}{y}}{y - 2z}}$$

Higher Order Derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = \underset{\text{read}}{\underrightarrow{f_{xy}}} = \underset{\text{read}}{\underleftarrow{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}} = \underset{\text{read}}{\underleftarrow{\frac{\partial^2 f}{\partial y \partial x}}} \quad \left(\begin{matrix}\text{first } x,\\ \text{then } y\end{matrix}\right)$$

$(f_y)_x$,

* second partials *

$(f_y)_y$

ex: find all second partials of $f(x,y) = x^4y^3 - y^4$

$$f_x = \frac{\partial f}{\partial x} = 4x^3y^3, \quad f_y = \frac{\partial f}{\partial y} = 3x^4y^2 - 4y^3$$

$$f_{xx} = \frac{\partial}{\partial x} f_x = 12x^2y^3$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = 12x^3y^2$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = 12x^3y^2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = 6x^4y - 12y^2$$

*Clairaut's Theorem:

$f_{xy} = f_{yx}$

(if both are continuous)

Keep going...

$$f_{xxyx} = \frac{\partial^4}{\partial x \partial y \partial x \partial x} f \ldots$$

Chain Rule:

interpretation as directional derivatives

$$\frac{\partial z}{\partial t_i}(\vec{a}) = \nabla f(\vec{G}(\vec{a})) \cdot \frac{\partial \vec{G}}{\partial t_i}(\vec{a})$$

$$* \quad \frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$$* \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Sum over all input variables of z

ex. Let $z = f(x,y) = x^2 + y^2 + xy$ and suppose $x = \sin t$, $y = e^t$. Find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = (2x+y)\cos t + (2y+x)e^t$$

$$= (2\sin t + e^t)\cos t + (2e^t + \sin t)e^t$$

or:

write $z = (\sin t)^2 + (e^t)^2 + (\sin t)e^t$.

find $\frac{dz}{dt}$.

ex> let $z = f(x,y) = x^2 - 2xy + y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Find z_r and z_θ

dependency tree:

$$\begin{tikzcd}[column sep=small]
 & & & z \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & x \arrow[dl, dash] \arrow[d, dash] & & y \arrow[d, dash] \arrow[dr, dash] & & \\
 & r & \theta & & r & \theta &
\end{tikzcd}$$

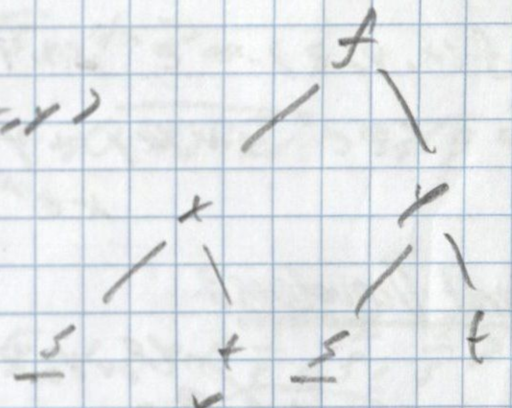
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}.$$

[Lecture Notes] Dependency Trees, Gradients and Directional Derivatives 9.18.23

Dependency Trees:

$f(x(s,t), y(s,t))$

depending tree:



$\mathbb{R}^2 \to \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$s, t \qquad x(s,t) \qquad z = f(x,y)$

$\qquad\quad y(s,t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

ex. $F(x,y) = f(\overbrace{g(x)k(y)}^{u}, \overbrace{g(x)+h(y)}^{v})$. Find $\frac{\partial F}{\partial x}$ & $\frac{\partial F}{\partial y}$

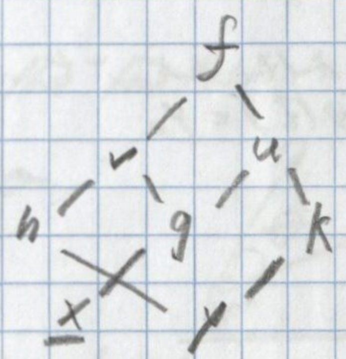
$\mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$

$x, y \qquad g(x) \qquad u = gk$

$\qquad\quad k(y)$

$\qquad\quad h(y) \qquad v = g+h$

$\qquad\qquad\qquad u(g,k)$



$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\boxed{\frac{\partial u}{\partial g}}\frac{dg}{dx} + \frac{\partial f}{\partial v}\boxed{\frac{\partial v}{\partial g}}\frac{dg}{dx}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\boxed{\frac{\partial u}{\partial k}}\frac{dk}{dy} + \frac{\partial f}{\partial v}\boxed{\frac{\partial v}{\partial h}}\frac{dh}{dy}$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}k(y)\frac{dg}{dx} + \frac{\partial f}{\partial v}(1)\frac{dg}{dx}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}g(x)\frac{dk}{dy} + \frac{\partial f}{\partial v}(1)\frac{dh}{dy}$$

Gradient:

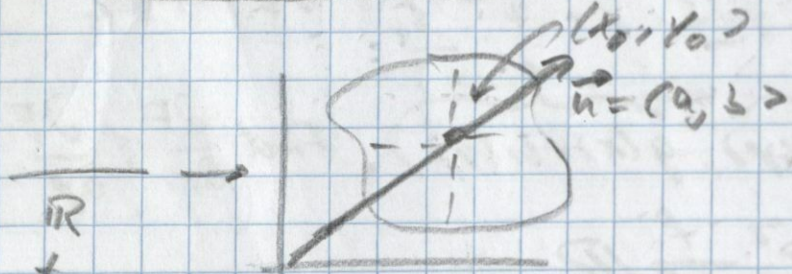
* $f(x,y) \longrightarrow \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \vec{\nabla} f$

* $f(x,y,z) \longrightarrow \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \vec{\nabla} f$

ex: $f(x,y,z) = e^{xz}\sin(xy)$

$$\vec{\nabla} f = \langle ze^{xz}\sin(xy) + ye^{xz}\cos(xy),\ xe^{xz}\cos(xy),\ xe^{xz}\sin(xy) \rangle.$$

Directional Derivatives:



$t \longmapsto x(t) = x_0 + t$
$y(t) = y_0$ $\longrightarrow f(x_0 + t, y_0)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x_0, y_0)$$

$t \longmapsto x(t) = x_0 + at$
$y(t) = y_0 + bt$ $\xrightarrow{f} \mathbb{R}$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

$$= \underbrace{\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle}_{\vec{\nabla} f} \cdot \underbrace{\langle a, b \rangle}_{\vec{u}}$$

Directional Derivative:

* $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$

$\vec{u}$ = unit vector!

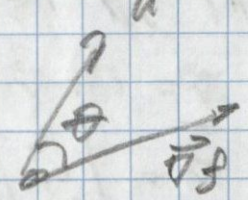
ex. f from last example
Compute directional derivative of f in direction of $\vec{v} = \begin{bmatrix}1\\2\\2\end{bmatrix}$ at $(x,y,z) = (2, \frac{\pi}{12}, 0)$.

1) $$\vec{\nabla} f(2, \tfrac{\pi}{2}, 0) = \langle \tfrac{\pi}{12}\cdot 1 \cdot \cos\tfrac{\pi}{6}, 2\cdot 1\cdot \cos\tfrac{\pi}{6}, 2\cdot 1 \cdot \sin\tfrac{\pi}{6} \rangle$$
$$= \langle \tfrac{\pi\sqrt{3}}{24}, \sqrt{3}, 1 \rangle.$$

2) $\vec{v} = \langle 1, 2, 2 \rangle \Rightarrow \|\vec{v}\| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$
$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle.$$

$$D_{\hat{v}} f(2, \tfrac{\pi}{12}, 0) = \underbrace{\langle \tfrac{\pi\sqrt{3}}{24}, \sqrt{3}, 1 \rangle}_{\vec{\nabla} f} \cdot \underbrace{\langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle}_{\hat{v}}$$
$$= \frac{\pi\sqrt{3}}{72} + \frac{2\sqrt{3}}{3} + \frac{2}{3}$$

Q. In what direction is f changing the fastest?



$$* \; D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \|\vec{\nabla} f\| \|\hat{u}\| \cos\theta$$
$$= \|\vec{\nabla} f\| \cos\theta$$

max'd at $\theta = 0°$ ie max'd in direction of $\vec{\nabla} f$!

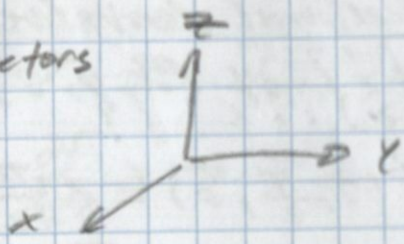
* $\|\vec{\nabla} f\|$ is the maximum rate of change!

ex. Compute maximum rate of change at $(2, \frac{\pi}{12}, 0)$.
(from prev ex.)

$$\|\vec{\nabla} f(2, \tfrac{\pi}{12}, 0)\| = \|\langle \tfrac{\pi\sqrt{3}}{24}, \sqrt{3}, 1 \rangle\|$$
$$= \sqrt{\left(\tfrac{\pi\sqrt{3}}{24}\right)^2 + (\sqrt{3})^2 + (1)^2}$$
$$= \sqrt{\frac{3\pi^2}{24^2} + 3 + 1}$$

Independent Notes | Exam 1: Standards 01-05   9.18.23

Standard 01: 3D Space and Vectors

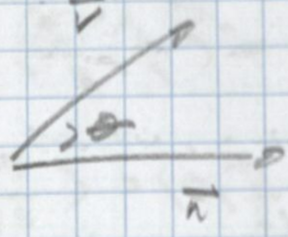


$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

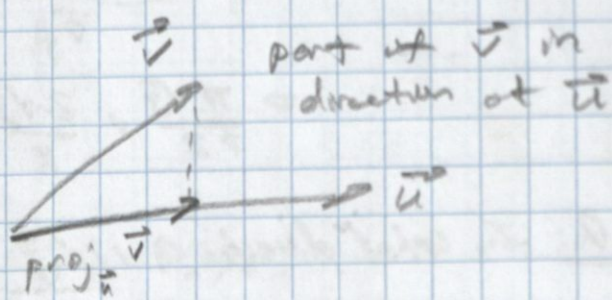
$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$



$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}$

$r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$



$\text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$

$\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}\right) \hat{u}$

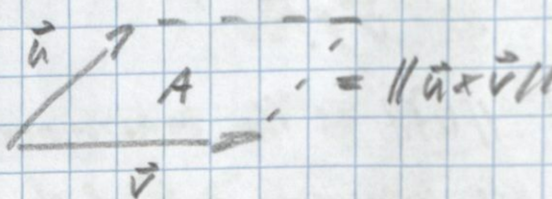
$\|\text{proj}_{\vec{u}} \vec{v}\| = \text{comp}_{\vec{u}} \vec{v}$

$\vec{w} = \vec{u} \times \vec{v}$

$\vec{w} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$\vec{u} \times \vec{v} = 0 \iff \vec{u} \parallel \vec{v}$



Standard 02: Lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

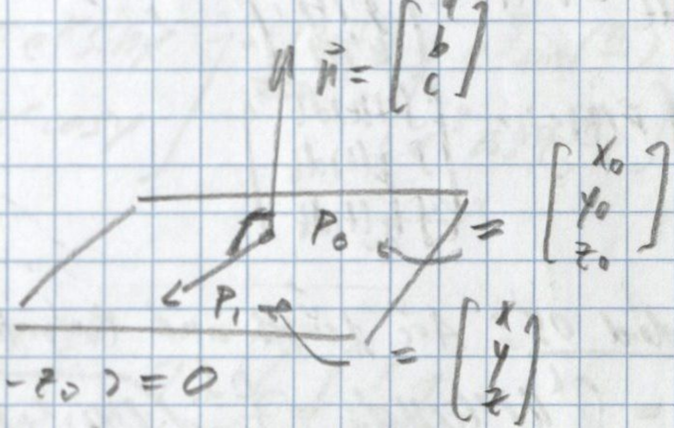
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

skew: non-parallel & non-intersecting

Standard 03: Planes

$$\vec{n} \cdot \overrightarrow{P_0 P_1} = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$



$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

2 lines: intersect, parallel, or skew

line and plane: parallel (line contained in plane, line never intersects plane), intersecting

$$ax + by + cz = d$$

Stadard 04: Vector Valued Functions

$$\vec{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

$$x^2 + y^2 = r^2 \iff \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}$$

$$\lim_{t \to a} \vec{r}(t) = \begin{bmatrix} \lim_{t \to a} f(t) \\ \lim_{t \to a} g(t) \\ \lim_{t \to a} h(t) \end{bmatrix}$$

2 planes intersect at line

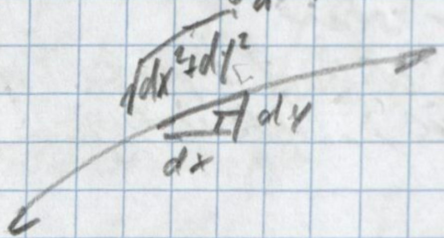
2 surfaces intersect at curve

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$$

$$\int \vec{r}(t)\,dt = \begin{bmatrix} \int f(t)\,dt \\ \int g(t)\,dt \\ \int h(t)\,dt \end{bmatrix}$$

Stadard 05: Arc length and Curvature?

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$$



$$\vec{r}(t) = \vec{r}(t(s)), \quad \underset{(t=a)}{0} \le s \le \underset{(t=b)}{L}$$

$$s = \int_a^t \|\vec{r}\,'(u)\|\,du$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

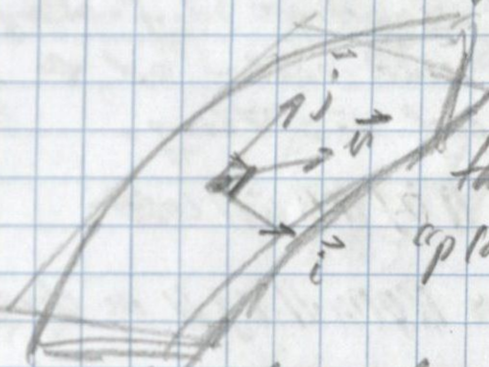
Lecture Notes

9.20.23

Directional Derivatives :

$* \; D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$

$z = f(x,y)$



tangent plane
"plane of best fit"

ex. find the directional derivative of $f(x,y) = e^x \sin y$ at $(3, \frac{\pi}{6})$ in the direction of $\vec{v} = \langle 1, -2 \rangle$

$$\vec{\nabla} f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} e^x \sin y \\ e^x \cos y \end{bmatrix} \overset{\text{at } (3, \frac{\pi}{6})}{=} \begin{bmatrix} e^3 \sin(\frac{\pi}{6}) \\ e^3 \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} e^3 \cdot \frac{1}{2} \\ e^3 \cdot \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}{\sqrt{5}}$$

$$D_{\hat{v}} f = \vec{\nabla} f \cdot \hat{v} = \begin{bmatrix} \frac{e^3}{2} \\ \frac{\sqrt{3}}{2} e^3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{e^3}{2\sqrt{5}} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{e^3}{2\sqrt{5}} (1 - 2\sqrt{3})$$

Gradient Application:

Q. In what direction is f changing the fastest?
Let $\vec{u}$ be unit vector and θ be the angle b/w ∇f and $\vec{u}$. Then,

* $D_{\vec{u}} f = \|\nabla f\| \|\vec{u}\| \cos\theta$ — happens when $\theta = 0$

* maximum ROC: $\|\nabla f\|$ in direction ∇f

* minimum ROC: $-\|\nabla f\|$ in direction $-\nabla f$

* gradient descent/ascent algorithms *

ex. Find the maximum ROC of $f(x,y,z) = \frac{x+y}{z}$ at $(1,1,-1)$ and the direction where it occurs

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{bmatrix} \frac{1}{z} \\ \frac{1}{z} \\ -\frac{(x+y)}{z^2} \end{bmatrix} \overset{\text{at } (1,1,-1)}{=} \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} = \text{direction of max rate of change}$$

max rate:

$$= \|\nabla f\| = \sqrt{(-1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}.$$

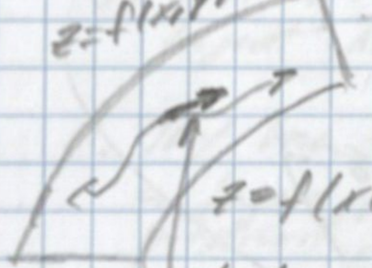
$$\vec{u} = \frac{1}{\|\nabla f\|} \nabla f = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$$

~~better~~ as a unit vector

* Gradient vector field: plot $\nabla f(x,y)$ at every point (x,y) *

Recall Chain Rule:

$z = f(x(t), y(t))$, then $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$



$$= \left(D_{\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}} f\right)\left(\left\| \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} \right\|\right)$$

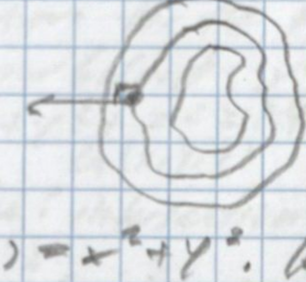
direction in (x, y) space is $\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}$, speed is $\left\| \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} \right\|$

Geometric Use of Gradient:

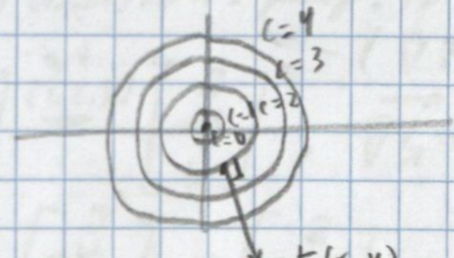
* ∇F is perpendicular to the level set $F = c$

In particular:

- $\nabla F(x,y)$ is perpendicular to the level curve $F(x,y) = c$ at $(x$
- $\nabla F(x,y,z)$ is a normal vector to the tangent plane of $F(x,y,z) = c$ at (x,y,z)



ex. $F(x,y) = x^2 + y^2$. level curves: $x^2 + y^2 = c$



ex.
Find the tangent plane and normal line to the surface $y = x^2 - z^2$ at $(4, 7, 3)$

Write $y = x^2 - z^2$ in the form $F(x, y, z) = c$

$$\underbrace{x^2 - z^2 - y}_{F(x,y,z)} = \underbrace{0}_{c}$$

normal to tangent plane

$$\nabla F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 2x \\ -1 \\ -2z \end{bmatrix} \overset{\text{at } (4,7,3)}{=} \begin{bmatrix} 8 \\ -1 \\ -6 \end{bmatrix}$$

$\rightsquigarrow 8x - y - 6z = d$
passes through $(4, 7, 3)$

$8(4) - 7 - 6(3) = d \Rightarrow$

tangent plane: $8x - y - 6z = 7$

normal line: perpendicular to tangent plane (and passing through point)

$$\ell(t) = \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} + t \begin{bmatrix} 8 \\ -1 \\ -6 \end{bmatrix}$$

ex: Show that the ellipsoid $\overbrace{3x^2+2y^2+z^2=9}^{F(x,y,z)}$ is tangent to the sphere $\underbrace{x^2+y^2+z^2-8x-6y-8z+24=0}_{G(x,y,z)}$ at $(1,1,2)$.

Q. What does it mean for two surfaces to be tangent at a point of intersection?

A. They have the same tangent planes. Since a tangent plane is perpendicular to the gradient, this means the gradients are parallel

$\parallel \vec{\nabla} F$

$$\nabla F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} 6x \\ 4y \\ 2z \end{bmatrix} \Bigg|_{\text{at } (1,1,2)} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}, \quad \nabla G = \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} 2x-8 \\ 2y-6 \\ 2z-8 \end{bmatrix} \Bigg|_{\text{at } (1,1,2)} = \begin{bmatrix} -6 \\ -4 \\ -4 \end{bmatrix}$$

$\nabla F \parallel \nabla G$!

ex. Find an equation for the tangent line to the intersection of the hyperboloid $\overbrace{x^2-y^2+z^2=6}^{F(x,y,z)}$ and the sphere $\underbrace{x^2+y^2+z^2=14}_{G(x,y,z)}$ at the point $(1,2,3)$.

tangent line: intersection of the 2 tangent planes

point: $(1,2,3)$ direction: perp. to both normal vectors $(\nabla F \times \nabla G)$

$$\vec{\nabla} F = \begin{bmatrix} 2x \\ -2y \\ 2z \end{bmatrix} \overset{\text{at } (1,2,3)}{=} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}, \quad \vec{\nabla} G = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \overset{\text{at } (1,2,3)}{=} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\vec{\nabla} F \times \vec{\nabla} G = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \propto \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ 4 \end{bmatrix}$$

$$L(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -12 \\ 0 \\ 4 \end{bmatrix}$$

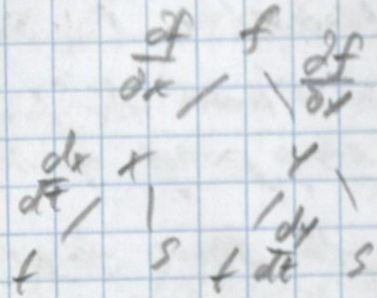
$\boxed{\text{Tutorial Notes}}$ 9.21.25

Partial Derivatives:

ex. $f(x,y) = \frac{1}{x^2+y^2}$

$x = t^2 + s$

$y = 2t - s$



$\frac{\partial f}{\partial t}(s=1,\ t=1)$

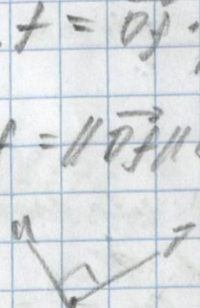
$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

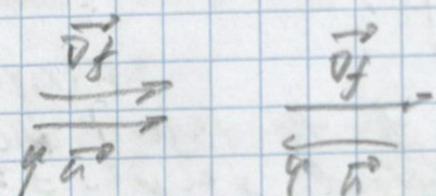
$= -\frac{2x}{(x^2+y^2)} \cdot (2t) + \frac{-2y}{(x^2+y^2)^2}(2)$

Directional Derivatives:

$D_{\vec{u}} f = \vec{\nabla} f \cdot \frac{\vec{u}}{\|\vec{u}\|}$ $\qquad \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$D_{\vec{u}} f = \|\vec{\nabla} f\| \cos\theta$





$D_{\vec{u}} f$ is max $\qquad D_{\vec{u}} f$ is min

ex. $e_1 = \langle 1, 0, 0 \rangle$

$D_{e_1} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \langle 1, 0, 0 \rangle$

$= \frac{\partial f}{\partial x}$

ex. Show that the paraboloid $2x^2+y^2-z=5$ and the sphere $(x-3)^2+(y-4)^2+(z-\frac{1}{2})^2=\frac{33}{4}$ are tangent to each other at $\langle 1,2,1\rangle$.
Find a plane tangent to both surfaces

$F(x,y,z) = 2x^2+y^2-z=5$

$G(x,y,z) = (x-3)^2+(y-4)^2+(z-\frac{1}{2})^2 = \frac{33}{4}$

$\vec{\nabla} F = \langle 4x, 2y, -1\rangle$

$\vec{\nabla} G = \langle 2x-6, 2y-8, 2z-1\rangle$

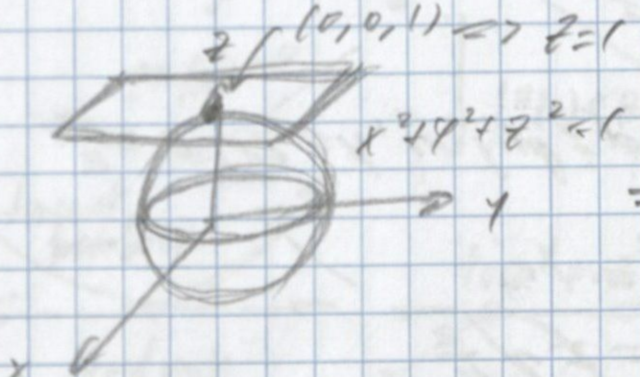
$\vec{\nabla} F(1,2,1) = \langle 4,4,-1\rangle$

$\vec{\nabla} G(1,2,1) = \langle -4,-4,1\rangle$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=d$

$\Rightarrow 4(x-1)+4(y-2)+(-1)(z-1)=0$

$\Rightarrow 4x+4y-z=11$

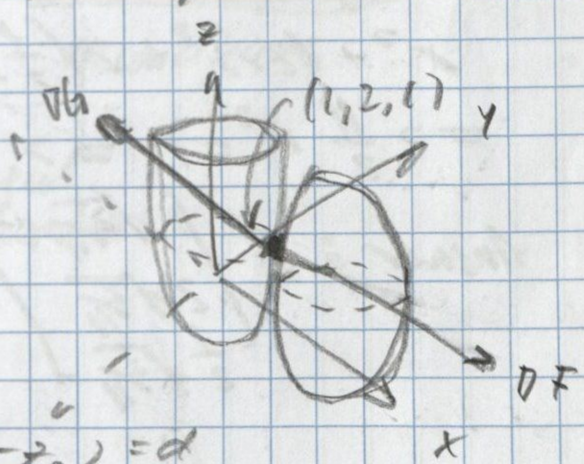


$a(x-x_0)+b(y-y_0)+c(z-z_0)=d$

$-4(x-1)-4(y-2)+1(z-1)=0$

$\Rightarrow -4x-4y+z=-11$

$\Rightarrow 4x+4y-z=11$



$f(x,y,z) = x^2+y^2+z^2-1$

$\nabla f = \langle 2x, 2y, 2z\rangle$

$\nabla f(0,0,1) = \langle 0,0,2\rangle$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=d$

$\Rightarrow 2(z-1)=0$

$\Rightarrow z-1=0$

$\Rightarrow z=1$

Lecture Notes

9.22.23

ex. At what point(s) on the ellipsoid $\overset{F(x,y,z)}{x^2+y^2+2z^2=1}$ is the tangent plane parallel to the plane $x+2y+z=1$

$$\nabla F = \begin{bmatrix} 2x \\ 2y \\ 4z \end{bmatrix}$$ = normal vector to tangent plane at (x,y,z)

$\Rightarrow$ needs to be parallel to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solve $\begin{bmatrix} 2x \\ 2y \\ 4z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 4z = \lambda \end{cases}$

$\Rightarrow \lambda = 2x = y = 4z$

symmetric equations for a line

intersect line with ellipsoid: plug in $y = 2x$, $z = \frac{x}{2}$ into ellipsoid:

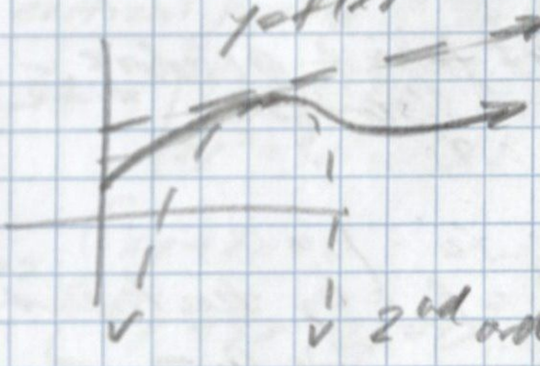
$$x^2 + (2x)^2 + 2\left(\frac{x}{2}\right)^2 = 1$$

$$\Rightarrow \frac{11}{2}x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{2}{11}}.$$

Answer: $\pm \begin{bmatrix} \sqrt{2/11} \\ 2\sqrt{2/11} \\ \frac{1}{2}\sqrt{2/11} \end{bmatrix}$

2^{nd} derivative test: $\mathbb{R}^2$ review

$f(x)$



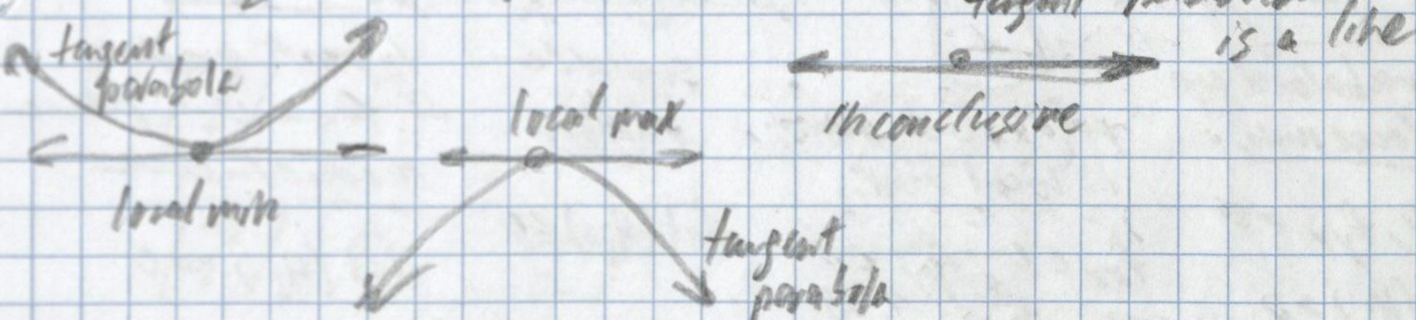
1^{st} order: tangent line

2^{nd} order: tangent parabola

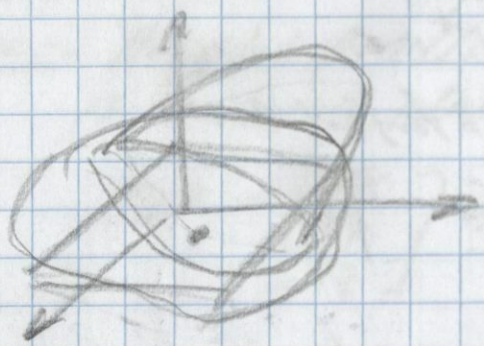
At a critical point, where $f'(a) = 0$, the tangent line is horizontal, and the tangent parabola looks like:

$$y = \frac{f''(a)}{2} x^2$$

$f''(a) > 0$ $\quad$ $f''(a) < 0$ $\quad$ $f''(a) = 0$



$\underline{f(x,y)} = \mathbb{R}^3$



1^{st} order: tangent plane

2^{nd} order: tangent quadric surface

At a critical point where $f_x = f_y = 0$, the tangent quadric surface looks like:

* $\underline{\text{Hessian}}$:

* $z = \frac{f_{xx}}{2} x^2 + f_{xy} xy + \frac{f_{yy}}{2} y^2$

* $H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

* $\det(H_f) = f_{xx} f_{yy} - f_{xy}^2$

paraboloid up
$\underline{\text{local min.}}$

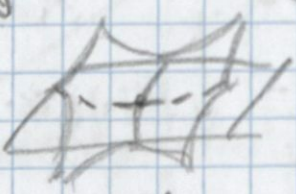
$>0, f_{yy} > 0$

et$(H_f) > 0$

paraboloid down
$\underline{\text{local max.}}$

$f_{xx} < 0, f_{yy} < 0$

$\det(H_f) > 0$



saddle
$\underline{\text{neither}}$

$\det(H_f) < 0$

tangent quadric surface is a plane
$\underline{\text{inconclusive}}$

$\det(H_f) = 0$

↑

discriminant $= -\det(H)$ ← why this is relevant

$(b^2 - 4ac)$

ex: Find and classify the critical points of
$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$$

critical points: $\vec{\nabla} f = \vec{0}$

$$\vec{\nabla} f = \begin{bmatrix} 6xy - 12x \\ 3y^2 + 3x^2 - 12y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} 6xy - 12x = 0 \\ 3y^2 + 3x^2 - 12y = 0 \end{cases}$$

$$\iff \begin{cases} xy - 2x = 0 & \rightsquigarrow \quad y = 2 \quad \text{or} \quad x = 0 \\ 3y^2 + 3x^2 - 12y = 0 & \rightsquigarrow \quad 4 + x^2 - 8 = 0 \quad \text{or} \quad y^2 - 4y = 0 \end{cases}$$

$$x = \pm 2 \qquad\qquad y = 4, 0$$

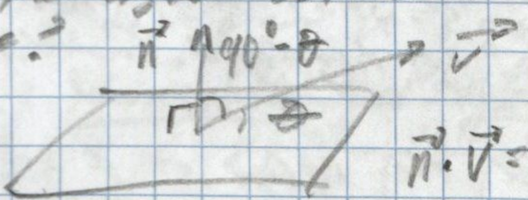
crit. point	f_{xx}	$\det(H(f))$	2nd derivative test
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	-12 $(-)$	144 $(+)$	local max
$\begin{bmatrix} 0 \\ 4 \end{bmatrix}$	12 $(+)$	144 $(+)$	local min
$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	0	-144 $(-)$	neither (saddle)
$\begin{bmatrix} -2 \\ 2 \end{bmatrix}$	0	-144 $(-)$	neither (saddle)

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6y - 12 & 6x \\ 6x & 6 \end{bmatrix}$$

$$\det(H(f)) = f_{xx} f_{yy} - f_{xy} f_{yx} = (6y - 12)^2 - (6x)^2$$

Q: How do we find the angle between a vector and a plane?

(webassign due Wed).



A. $\vec{n} \cdot \vec{v} = \|\vec{n}\| \|\vec{v}\| \cos(90° - \theta)$

Lecture Notes

9.25.23

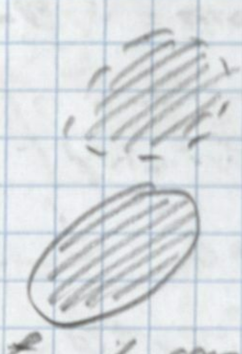
* Closed sets are sets which contain all of their boundary points

eg. $x^2+y^2<1$

not closed =>

eg. $x^2+y^2\le 1$

closed =>



* A set is bounded in $\mathbb{R}^2$ if it can fit in a disk

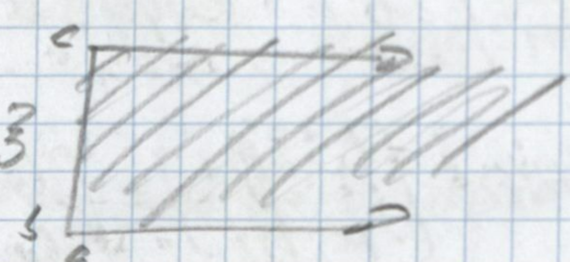
eg.

closed and bounded

a set of points bounded by polygon P is bound by the disk defined by the all points $x^2+y^2\le 9$

* Unbounded Example:

$\{(x,y) \mid x\ge a,\ b\le y\le c\}$



closed, but not bounded

* Finding extrema on closed and bounded sets:

* To find the absolute max and min values of a continuous function $f=f(x,y)$ on a closed and bounded set D:

1. Find the values of f at critical points of f in D
2. find the extreme values of f on boundary (parametrize
3. the largest value from 1. and 2. is abs max boundary curve) and lowest is abs min.

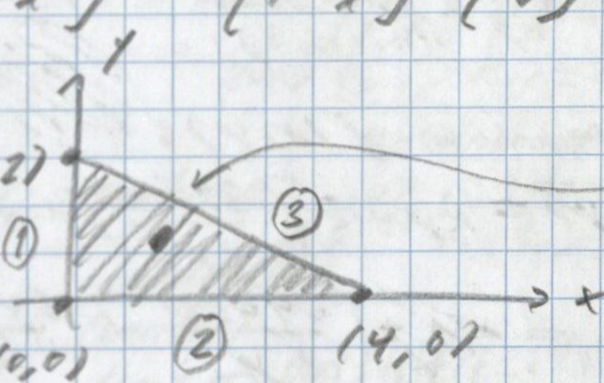
ex 1. Find the abs max and min of $f(x,y) = x+y-xy$ on the closed and bounded set D which is the closed triangle w/ vertices $(0,0)$, $(0,2)$, $(4,0)$.

1. <u>Critical points</u>: $\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (2nd derivative test is optional!)

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 1-y \\ 1-x \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1-y \\ 1-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is in domain

2. <u>Boundary</u>:



① $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $0 \le t \le 2$. $g_1(t) = f(0,t) = t$

critical points: $g_1'(t) = 0 \rightsquigarrow 1 = 0$

never happens, so no critical points!

endpoints: $t = 0, 2 \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

② $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$, $0 \le t \le 4$. $g_2(t) = f(t,0) = t$

critical points: $g_2'(t) = 0 \rightsquigarrow 1 = 0 \rightsquigarrow$ no critical points!

endpoints: $t = 0, 4 \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

③ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ 2-t \end{bmatrix} \left(= \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{\text{point}} + t \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{direction vector}} \right)$, $0 \le t \le 2$

$g_3(t) = f(2t, 2-t) = 2t + (2-t) - (2t)(2-t)$

$= 2t^2 - 3t + 2$

$g_3'(t) = 4t - 3$

critical points: $g_3'(t) = 0 \rightsquigarrow 4t - 3 = 0 \rightsquigarrow t = \frac{3}{4}$

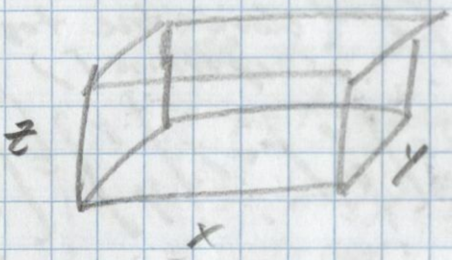
$\rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix}$

Endpoints: $\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

3. <u>Plug in Points</u>:

point	value of f	
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	1	
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	0	min.
$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	2	
$\begin{bmatrix} 4 \\ 0 \end{bmatrix}$	4	max.
$\begin{bmatrix} 3/2 \\ 5/4 \end{bmatrix}$	$\frac{7}{8}$	

ex. A cardboard box without a lid is to have a volume of 32,000 cm³. Find the dimensions of the box which uses the least amount of cardboard



volume (in cm³) $= 32{,}000 = xyz$

$\Rightarrow z = \dfrac{32000}{xy}$

area of cardboard (in cm³) $= xy + 2xz + 2yz = f(x,y)$

Want to minimize $f(x,y)$:

$$f(x,y) = xy + 2\left(\frac{32000}{xy}\right)x + 2\left(\frac{32000}{xy}\right)y$$

$$= xy + \frac{64000}{y} + \frac{64000}{x}, \quad x, y > 0$$

neither closed or bounded

Observe $f \to \infty$ when $x \to 0^+$, $y \to 0^+$, $x \to \infty$, or $y \to \infty$.

So f has an absolute min at a critical point.

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} y - \frac{64000}{x^2} \\ x - \frac{64000}{y^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow y = \frac{64000}{x^2}$$

plug in

$$x - \frac{64000}{\left(\frac{64000}{x^2}\right)^2} = 0$$

$$x\left(\frac{64000}{x^2}\right)^2 = 64000$$

$$\cancel{x} \cdot \frac{64000^{\cancel{2}}}{x^{\cancel{4}3}} = \cancel{64000} \rightsquigarrow x^3 = 64000$$

$$x = 40$$

$$y = \frac{64000}{x^2} = \frac{64000}{40^2} = 40$$

symmetrical functions

then $32000 = 40 \cdot 40 \cdot z \Rightarrow z = 20$

minimum when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}$

so 40cm × 40cm × 20cm box

[Lecture Notes] 9.27.23

ex: Find the local min/max of $f = x^4 + y^4 - 4xy + 1$

critical points: $\nabla f = \vec{0}$

$$\nabla f = \begin{bmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x^3 = y \quad \text{plug in}$$

$$4(x^3)^3 - 4x = 0$$

$$\iff 4x(x^8 - 1) = 0 \Rightarrow x = 0, 1, -1$$

$$\Rightarrow |x| = 1$$

2nd derivative test:

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

$$\det(H(f)) = 144x^2y^2 - 16$$

critical point	$\det(H(f))$	f_{xx}	conclusion
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	-16 (−)		saddle (neither)
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	128 (+)	12 (+)	local min
$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	128 (+)	12 (+)	local min

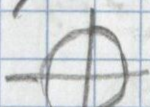
ex: Find the absolute max and min of $f(x,y) = x^2 + y^2 + y$ (continuous) on the disk $x^2 + y^2 \leq 1$ (closed + bounded) and the point(s) where the extrema are achieved.

1. Critical Points: $\nabla f = \vec{0}$

$$\nabla f = \begin{bmatrix} 2x \\ 2y + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{cases} 2x = 0 \\ 2y + 1 = 0 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

check $x^2 + y^2 \leq 1$ ✓

2. Parameterize boundary: $x^2 + y^2 = 1$



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad 0 \leq t \leq 2\pi$$

or use Lagrange multipliers

$$g(t) = f(\cos t, \sin t) = (\cos t)^2 + (\sin t)^2 + \sin t = 1 + \sin t$$

3. Find critical points of g: $g'(t) = 0$

$$g'(t) = \cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

4. Test points:

critical point	value at f	
$\begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$	$0 + (-\frac{1}{2})^2 + (-\frac{1}{2}) = -\frac{1}{4}$	← abs min
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	2	← abs max
$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$	0	

Lagrange Multipliers:

want to extremize $f(x,y,z)$ subject to constraint $g(x,y,z)=k$, a level surface of g

* $\nabla f(P) \parallel \nabla g(P)$. So $\nabla f(P) = \lambda \nabla g(P)$.
* λ is called the Lagrange Multiplier

* Why? Recall a critical point of f is when

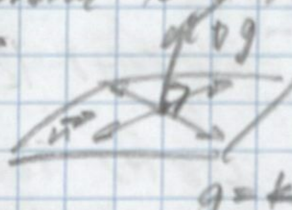
"equivalently" $D_{\vec{u}} f = 0$ for all unit vectors $\vec{u}$

"if and only if" $\iff \nabla f \cdot \vec{u} = 0$ for all unit vectors $\vec{u}$

$\iff \nabla f = \vec{0}$.

Now, a critical point of f on the level set $g=k$ is when $D_{\vec{u}} f = 0$ for all unit vectors $\vec{u}$ parallel to $g=k$

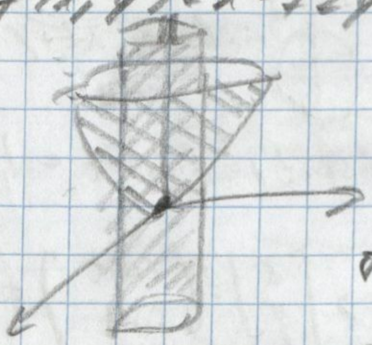
$\iff \nabla f \cdot \vec{u} = 0$ for all unit vectors $\vec{u}$ perpendicular to ∇g



$\iff \nabla f$ and ∇g are parallel

* $\begin{cases} \nabla f(x,y,z) = \lambda g(x,y,z) \\ g(x,y,z) = k \end{cases} \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = k \end{cases}$

ex: Find the extreme values of $f(x,y) = x^2 + 2y^2$ subject to the constraint $\underbrace{x^2+y^2}_{g} = 1$



find maximum values of f when constrained (or intersected) w/ cylinder g

1. Find where $\nabla f = \lambda \nabla g$ and $g = 1$

$\nabla f = \lambda \nabla g \iff \begin{bmatrix} 2x \\ 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \iff \begin{cases} 2x = \lambda 2x \Rightarrow x = \lambda x \\ 4y = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$

$x = \lambda x$

$x = 0$
$x^2+y^2 = 1 \Rightarrow y = \pm 1$
$\lambda = 2$

$x \neq 0$, $\lambda = 1$
$4y = \lambda \cdot 2y$
$y = 0$
$\Rightarrow x = \pm 1$

2. Test points:

point	value of f		point	value of f	
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	2	abs maxima	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	1	abs minima
$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$	2		$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	1	

|Lecture Notes| 9.29.23

ex. Find the extrema values of $f(x,y) = e^{-xy}$ on the region $x^2+4y^2 \le 1$.

(1) Critical points:

closed + bounded

$$\nabla f = \begin{bmatrix} -ye^{-xy} \\ -xe^{-xy} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y=0, \Rightarrow x=0 \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0^2 + 4\cdot 0^2 \le 1$ ✓

(2) Boundary: Lagrange mults.

$$\begin{cases} \nabla f = \lambda \nabla g \\ x^2+y^2=1 \end{cases} \rightsquigarrow \begin{bmatrix} -ye^{-xy} \\ -xe^{-xy} \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 8y \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} ① \; -ye^{-xy} = \lambda \cdot 2x \rightsquigarrow e^{-xy} = \frac{\lambda \cdot 2x}{-y} \\ ② \; -xe^{-xy} = \lambda \cdot 8y \rightsquigarrow x \cdot \left(\frac{\lambda 2x}{-y}\right) = \lambda 8y \\ ③ \; x^2+4y^2=1 \end{cases}$$

$\Rightarrow 2x^2 = 8y^2$

$x^2 = 4y^2$

③ $\Rightarrow x^2 + x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$y^2 = \frac{1}{4}x^2 \Rightarrow y = \frac{1}{2}(\pm x) = \pm \frac{1}{2\sqrt{2}}$

$\leftarrow$ ①

$y=0, \lambda=0$

② $\rightarrow x=0$ doesn't

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfy ③

(2') Parameterize boundary $x^2+4y^2=1$

$\cos^2 t + \sin^2 t = 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \frac{1}{2}\sin t \end{bmatrix}, \; 0 \le t \le 2\pi$$

$h(t) = f(\cos t, \frac{1}{2}\sin t)$

$= e^{-\frac{1}{2}\cos t \sin t} = e^{-\frac{\sin 2t}{4}}$

(no endpoints)

Critical points $h'(t)=0$:

$$-\frac{1}{2}\cos(2t)\, e^{-\frac{\sin(2t)}{4}} = 0$$

$\Leftrightarrow \cos(2t) = 0$

$\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(3) Test Points

point	value of $f = e^{-xy}$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$1 = e^0$
$\begin{bmatrix} 1/\sqrt{2} \\ 1/2\sqrt{2} \end{bmatrix}$	$e^{-\frac{1}{4}}$
$\begin{bmatrix} 1/\sqrt{2} \\ -1/2\sqrt{2} \end{bmatrix}$	$e^{\frac{1}{4}}$
$\begin{bmatrix} -1/\sqrt{2} \\ 1/2\sqrt{2} \end{bmatrix}$	$e^{\frac{1}{4}}$
$\begin{bmatrix} -1/\sqrt{2} \\ -1/2\sqrt{2} \end{bmatrix}$	$e^{-\frac{1}{4}}$

absolute maxima

absolute minima

Ex: Find the minimum value of $f(x,y,z) = x^2+y^2+z^2$ with respect to the constraint $x^2yz=2$ and the point(s) where the minimum value is achieved.

"$x^2yz=2$ is closed + unbounded, but $f \to \infty$ when $x \to \pm\infty$, $y \to \pm\infty$, or $z \to \pm\infty$. So absolute minimum exists."

Lagrange Multipliers: $\begin{cases} \nabla f = \lambda \nabla g \\ x^2yz = 2 \end{cases}$ $\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$, $\nabla g = \begin{bmatrix} 2xyz \\ x^2z \\ x^2y \end{bmatrix}$

$$\Leftrightarrow \begin{cases} 2x = 2\lambda xyz \\ 2y = \lambda x^2 z \\ 2z = \lambda x^2 y \\ x^2yz = 2 \end{cases} \underset{x,y,z\neq 0}{\Rightarrow} \quad \overset{\text{using } x^2yz=2}{\Longleftrightarrow} \begin{cases} 2x = 2\lambda\left(\frac{2}{x}\right) \\ 2y = \lambda\left(\frac{2}{y}\right) \\ 2z = \lambda\left(\frac{2}{z}\right) \\ x^2yz = 2 \end{cases} \Leftrightarrow \begin{cases} x^2 = 2\lambda \\ y^2 = 2\lambda \quad (\Rightarrow \lambda \geq 0) \\ z^2 = 2\lambda \\ x^2yz = 2 \end{cases}$$

$x^2yz = 2 \Rightarrow (2\lambda)\cdot(\pm\sqrt{\lambda})\cdot(\pm\sqrt{\lambda}) = 2$

$\Rightarrow 2\lambda^2 = 2 \Rightarrow \lambda = \pm 1$

$$\Leftrightarrow \begin{cases} x^2 = 2 \\ y^2 = 1 \\ z^2 = 1 \\ x^2yz = 2 \end{cases} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \pm\begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}, \pm\begin{bmatrix} \sqrt{2} \\ -1 \\ -1 \end{bmatrix}$$

Test Points: $f = x^2+y^2+z^2 = 2+1+1 = 4$.

<u>All</u> are absolute minima

"b/c by initial statement there <u>has</u> to be an absolute minima, and all the critical points have the same value, all of the critical points are absolute minima."

<u>or</u>, $x^2 = \frac{2}{yz}$, minimize $\underbrace{\left(\frac{2}{yz}\right)^2 + y^2 + z^2}_{\nabla F = \vec{0}}$ over (y,z).

Lagrange Multipliers w/ 2 constraints:

* extremize $f(x,y,z)$ subject to $g(x,y,z)=c$ and $h(x,y,z)=k$.

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$$

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) & (\lambda, \mu \text{ scalars}) \\ g(x,y,z) = c & (\infty \text{ scalars possible}) \\ h(x,y,z) = k \end{cases}$$

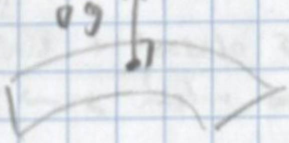
* a critical point of f on the intersection of the level sets $g=c$ and $h=k$ is where:

$D_{\vec{u}} f = 0$ for all unit vectors $\vec{u}$ parallel to $g=c$ and $h=k$

$\Leftrightarrow \nabla f \cdot \vec{u} = 0$ for all unit vectors $\vec{u}$ perpendicular to ∇g and ∇h

$\Leftrightarrow \nabla f$ is in the plane spanned by ∇g and ∇h

$g=c$



$\Leftrightarrow \nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars λ, μ.

ex. Find the points on the conic section $z^2 = x^2+y^2$ and $z = x+y+2$ which are closest to the origin. $\underbrace{x^2+y^2-z^2=0}_{g}$ $\underbrace{x+y-z=-2}_{h}$

1. Want to minimize $f(x,y,z) = x^2+y^2+z^2$.
2. Lagrange Multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = -2 \end{cases} \Leftrightarrow \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

* for LM in 3 variables with 2 constraints, one additional constraint allows us to solve for x, y, z & i.e. ignore λ, μ

$$\Leftrightarrow \begin{cases} ① & 2x = 2\lambda x + \mu \\ ② & 2y = 2\lambda y + \mu \\ ③ & 2z = -2\lambda z - \mu \\ ④ & x^2+y^2-z^2 = 0 \\ ⑤ & x+y-z = -2 \end{cases}$$

$① - ② = 2x - 2y = 2\lambda x - 2\lambda y$

$2(x-y) = 2\lambda(x-y)$

$(2-2\lambda)(x-y) = 0$

$\underline{\lambda = 1}$ or $\underline{x = y}$

$\lambda = 1$:

$① \Rightarrow \mu = 0$

$③ \Rightarrow 2z = -2z$

$\Rightarrow z = 0$

$④ \Rightarrow x^2+y^2 = 0$

$\Rightarrow x+y = 0$

$⑤ \Rightarrow 0+0-0 = -2$

impossible!

(no solutions)

$x = y$:

$④, ⑤ \rightsquigarrow \begin{cases} 2x^2 - z^2 = 0 \\ 2x - z = -2 \end{cases}$

$\downarrow$

$z = 2x+2$

plug into ④: $z^2 = 2x^2$

$(2x+2)^2 = 2x^2$

$4x^2+8x+4 = 2x^2$

$x^2+4x+2 = 0$

Quadratic formula: $x = -2 \pm \sqrt{2}$

point:	value of f:
$x=y=-2+\sqrt{2}$, $z=-2+2\sqrt{2}$	$24-16\sqrt{2}$ min.
$x=y=-2-\sqrt{2}$, $z=-2-2\sqrt{2}$	$24+16\sqrt{2}$

extra ex: Use Lagrange multipliers to find the point on the plane $2x-y+3z+14=0$ closest to the origin.

i.e. want to minimize

$$f(x,y,z) = \begin{pmatrix}\text{distance from}\\ (x,y,z) \text{ to origin}\end{pmatrix}^2 = x^2+y^2+z^2$$

Lagrange Multipliers:

$$\begin{cases} \nabla f = \lambda \nabla g \\ 2x-y+3z+14=0 \end{cases} \longrightarrow \begin{bmatrix}2x\\2y\\2z\end{bmatrix} = \lambda\begin{bmatrix}2\\-1\\3\end{bmatrix}$$

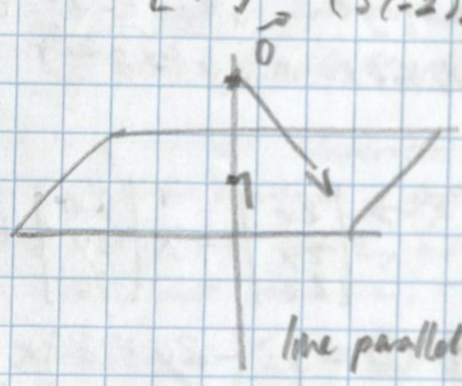
$$\Leftrightarrow \begin{cases} 2x=2\lambda \\ 2y=-\lambda \\ 2z=3\lambda \\ 2x-y+3z+14=0 \end{cases} \rightsquigarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\lambda\\-\lambda/2\\3\lambda/2\end{bmatrix}$$

$$\rightsquigarrow 2(\lambda) - \left(-\frac{\lambda}{2}\right) + 3\left(\frac{3\lambda}{2}\right) + 14 = 0$$

* g is closed + unbounded, but minimum exists since $f \to \infty$ when $x \to \pm\infty$, $y \to \pm\infty$, or $z \to \pm\infty$. *

$$\Rightarrow \lambda\left(2+\frac{1}{2}+\frac{9}{2}\right) = -14 \Rightarrow \lambda\left(\frac{14}{2}\right) = -14 \Rightarrow \lambda = -2$$

$\therefore$ minimized at $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2\\-(-2)/2\\3(-2)/2\end{bmatrix} = \begin{bmatrix}-2\\1\\-3\end{bmatrix}$



line parallel to $\begin{bmatrix}2\\-1\\3\end{bmatrix}$

Lecture Notes + last 2 examples 10.2.23

ex. find the absolute maximum and absolute minimum values of $f(x,y,z) = 2x+y$ with respect to the constraints $g(x,y,z) = 2x^2+z^2=4$ and $h(x,y,z) = 2x+y+3z=6$ and the point(s) where these extreme values are achieved.

Lagrange Multipliers!

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 4 \\ h = 6 \end{cases} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 4x \\ 0 \\ 2z \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 2 = 4\lambda x + 2\mu \longrightarrow 2 = 4\lambda x + 2 \rightarrow 4\lambda x = 0 \Rightarrow \boxed{x=0} \\ 1 = \mu \longrightarrow \mu = 1 \\ 0 = 2\lambda z + 3\mu \longrightarrow 0 = 2\lambda z + 3 \Rightarrow \lambda \neq 0 \\ 2x^2 + z^2 = 4 \\ 2x + y + 3z = 6 \end{cases}$$

plug in

$$\Leftrightarrow \begin{cases} z^2 = 4 \longrightarrow z = \pm 2 \\ y + 3z = 6 \rightarrow y = 6 - 3z \end{cases}$$

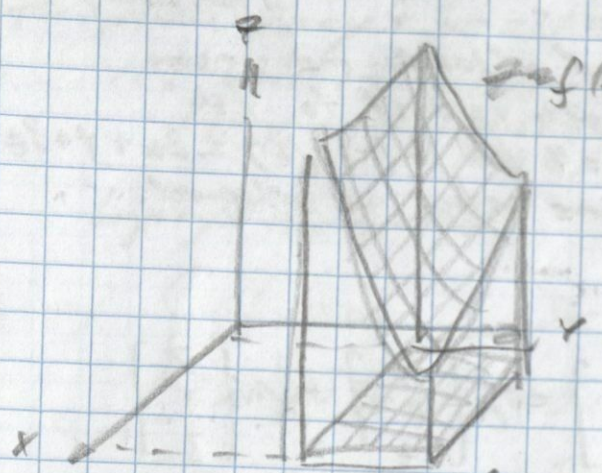
crit. point? value of f:

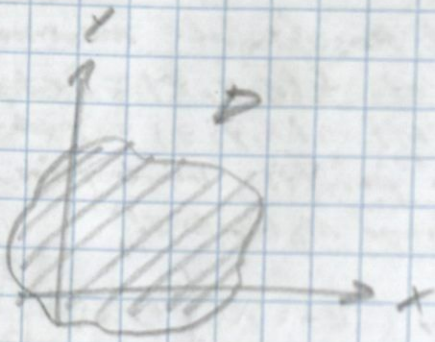
$x=0,\ y=0,\ z=2$ $\quad 0 \leftarrow$ absolute minimum

$x=0,\ y=12,\ z=-2$ $\quad 12 \leftarrow$ absolute maximum

Lecture Notes

10.4.23





$$= \iint_D f(x,y)\,dA$$ ← w/respect to area

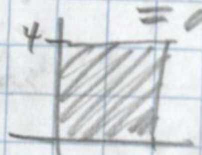
$$v = \sum_{i=1}^{m}\sum_{j=1}^{n}\Big(f(x_i,y_j)\,\overbrace{\Delta x\Delta y}^{\Delta A}\Big)$$

$m,n \to \infty$

= signed volume underneath $z=f(x,y)$ and above D

= area(D) · (average value of $f(x,y)$ on D)

ex: $$\int_0^2\int_0^4 y^3 e^{2x}\,dy\,dx$$



Fubini's Theorem

$$= \int_0^2\left(\int_0^4 y^3 e^{2x}\,dy\right)dx = \int_0^4\left(\int_0^2 y^3 e^{2x}\,dx\right)dy$$

2nd 1st 1st 2nd

1. $0 \le y \le 4$
2. $0 \le x \le 2$

1. $0 \le x \le 2$
2. $0 \le y \le 4$

$$= \int_{x=0}^{x=2}\left(\int_{y=0}^{y=4} y^3 e^{2x}\,dy\right)dx \qquad = \int_{y=0}^{y=4}\left(\int_{x=0}^{x=2} y^3 e^{2x}\,dx\right)dy$$

$$= \int_{x=0}^{x=2}\left[\frac{1}{4}y^4 e^{2x}\right]_{y=0}^{y=4}dx \qquad = \int_{y=0}^{y=4}\left[\frac{1}{2}y^3 e^{2x}\right]_{x=0}^{x=2}dy$$

$$= \int_{x=0}^{x=2}\left(\frac{1}{4}4^4 e^{2x} - \frac{1}{4}0^4 e^{2x}\right)dx \qquad = \int_{y=0}^{y=4}\frac{1}{2}y^3(e^4-1)\,dy$$

$$= \int_{x=0}^{x=2} 64e^{2x}\,dx \qquad = \left[\frac{e^4-1}{8}y^4\right]_{y=0}^{y=4}$$

$$= \left[32e^{2x}\right]_{x=0}^{x=2} \qquad = \frac{e^4-1}{8}(4^4-0^4)$$

$$= 32(e^4-1) \qquad = 32(e^4-1)$$

ex. compute $\iint_R ye^{-xy}\,dA$ where $R = [0,2]\times[0,3]$ ($0 \le y \le 3$, $0 \le x \le 2$)

$$= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2} ye^{-xy}\,dx\right)dy$$

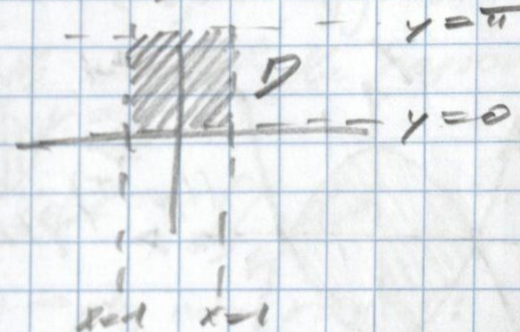
↓ substitute $u = -xy \Rightarrow \frac{du}{dx} = -y \Rightarrow \int -e^u\,du$

$$= \int_{y=0}^{y=3}\Big[-e^{-xy}\Big]_{x=0}^{x=2} dy = \int_{y=0}^{y=3}(-e^{-2y}+1)\,dy$$

$$= \left[\frac{e^{-2y}}{2}+y\right]_{y=0}^{y=3} = \left(\frac{e^{-6}}{2}+3\right)-\left(\frac{e^0}{2}+0\right) = \frac{e^{-6}}{2}+\frac{5}{2}.$$

ex. Find the volume of the solid bounded by the surface $z = \underbrace{1+e^x\sin y}_{\text{always pos on } D}$ and the planes $x=1$, $x=-1$, $y=0$, $y=\pi$, and $z=0$

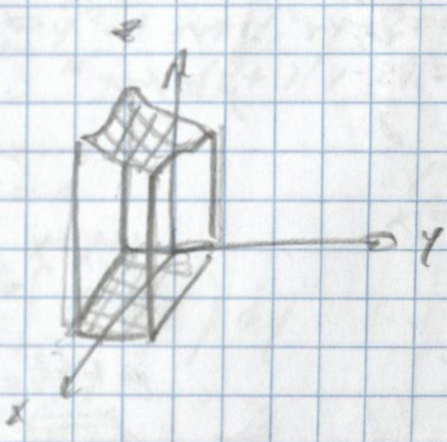
$-1 \le x \le 1$, $0 \le y \le \pi$



$$\therefore V = \iint_{\substack{-1\le x\le 1\\ 0\le y\le\pi}} (1+e^x\sin y)\,dA$$

$$= \int_{y=0}^{y=\pi}\left(\int_{x=0}^{x=1}(1+e^x\sin y)\,dx\right)dy$$

$$= \int_{y=0}^{y=\pi}\Big[x+e^x\sin y\Big]_{x=-1}^{x=1}dy$$



$$= \int_{y=0}^{y=\pi}\left(2+\left(e-\tfrac{1}{e}\right)\sin y\right)dy$$

$$= \left[2y-\left(e-\tfrac{1}{e}\right)\cos y\right]_{y=0}^{y=\pi}$$

$$= \left(2\pi-\left(e-\tfrac{1}{e}\right)\cos\pi\right)-\left(2\cdot 0-\left(e-\tfrac{1}{e}\right)\cos 0\right)$$

$$= 2\pi+2\left(e-\tfrac{1}{e}\right).$$

Slices 3 (non-rectangular regions of integration)

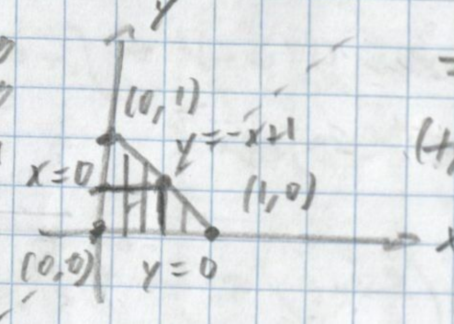
compute $\iint_D 5\,dA$ where D is the triangular region

$x \ge 0$

$y \ge 0$

$y \le -x+1$

$\Leftrightarrow x \le -y+1$



$\Rightarrow$  $\displaystyle\int_{x=0}^{x=1}\left(\int_{y=0}^{y=-x+1} f(x,y)\,dy\right)dx$

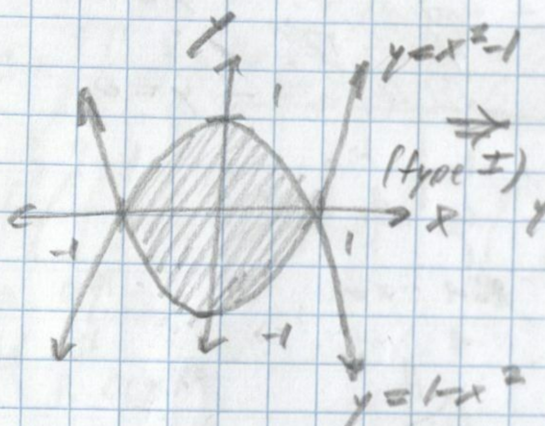
(type I)

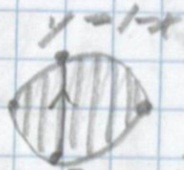
can depend on x

$\Rightarrow$  $\displaystyle\int_{y=0}^{y=1}\left(\int_{x=0}^{x=-y+1} f(x,y)\,dx\right)dy$

(type II)

Ex Compute $\iint_D x\,dA$ where D is the region bounded by the parabolas $y=1-x^2$ and $y=x^2-1$



$\Rightarrow$ 

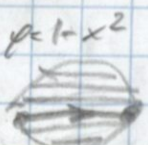
(type I)

$$\int_{x=-1}^{x=1}\left(\int_{y=x^2-1}^{y=1-x^2} x\,dy\right)dx$$

$$= \int_{x=-1}^{x=1}\left[xy\right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{x=-1}^{x=1} x(1-x^2) - x(x^2-1)\,dx$$

$$= \int_{x=-1}^{x=1} 2x - 2x^3\,dx = \left[x^2 - \tfrac{1}{2}x^4\right]_{x=-1}^{x=1} = 0.$$

$\Rightarrow$ 

(type II)

$y=x^2+1$

$x=\pm\sqrt{y+1}$

$$\iint_D x\,dA =$$

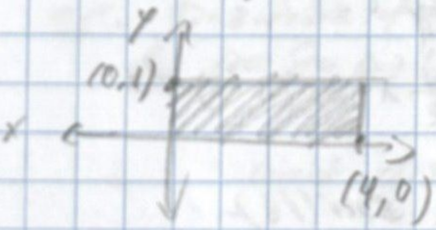
$$\int_{y=-1}^{y=0}\left(\int_{x=-\sqrt{y+1}}^{x=\sqrt{y+1}} dx\right)dy + \int_{y=0}^{y=1}\left(\int_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} dx\right)dy$$

$\boxed{\text{Lecture Notes}}$ 10.6.23

ex. Find the average of $f = e^y\sqrt{x+e^y}$ over the rectangle $[0,4]\times[0,1]$

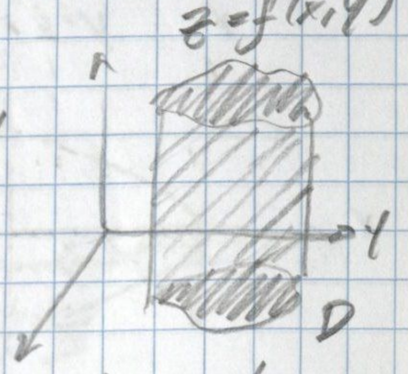
$0\le x\le 4 \quad 0\le y\le 1$

average value of f on D ($\bar{f}(x,y)$) $= \dfrac{1}{\text{area}(D)}\displaystyle\iint_D f(x,y)\,dA$



$$= \frac{1}{4}\iint_{\substack{0\le x\le 4\\ 0\le y\le 1}} e^y\sqrt{x+e^y}\,dA$$

$$= \dots = \frac{1}{15}\left[(4+e)^{\frac{5}{2}} - 5^{\frac{5}{2}} - e^{\frac{5}{2}} + 1\right]$$

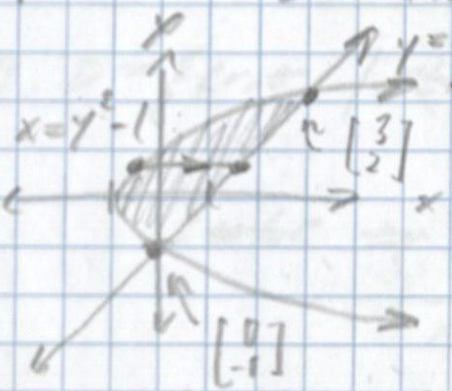


* Volume = base area × height

$\iint = \text{area}(D)\cdot\text{average}$

ex. Set up the integral to find the area of the region bounded by $x=y^2-1$ and $y=x-1$

* area of $D = A(D) = \iint_D 1\,dA = \iint_D dA$



$y=x-1 \Leftrightarrow x=y+1 \to$ plug into $x=y^2-1$:

$y+1=y^2-1 \leadsto y^2-y-2=0$

$(y+1)(y-2)=0.$

$y=-1$, $x=0$ $\qquad$ $y=2$, $x=3$

Type II $\to \displaystyle\int_{y=-1}^{y=2}\left(\int_{x=y^2-1}^{x=y+1} 1\,dx\right)dy$

$$= \int_{y=-1}^{y=2}\Big[x\Big]_{x=y^2-1}^{y+1}dy = \int_{-1}^{2}\left[(y+1)-(y^2-1)\right]dy$$

$$= \int_{-1}^{2}(-y^2+y+2)\,dy = \left[-\frac{1}{3}y^3+\frac{1}{2}y^2+2y\right]_{-1}^{2}$$

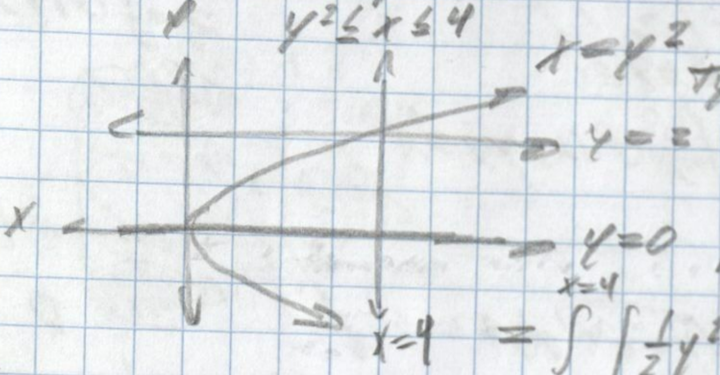
$\to \dfrac{9}{2}$

Ex: Compute $\int_0^2\int_{y^2}^4 y\cos(x^2)\,dx\,dy$.

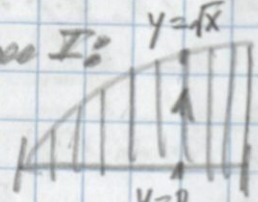
can't integrate with respect to x!

$0 \le y \le 2$

$y^2 \le x \le 4$



Type II:

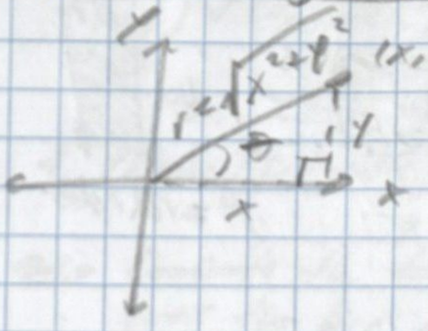


$$\int_{x=0}^{x=4}\left(\int_{y=0}^{y=\sqrt{x}} y\cos(x^2)\,dy\right)dx$$

$$= \int_{x=0}^{x=4}\left[\frac{1}{2}y^2\cos(x^2)\right]_{y=0}^{y=\sqrt{x}} = \int_{x=0}^{x=4}\frac{1}{2}x\cos(x^2)\,dx$$

$$\Rightarrow \begin{matrix} u = x^2 \\ du = 2x\,dx \end{matrix} \Rightarrow \frac{1}{4}\int \cos u\,du \Rightarrow \left[\frac{\sin(x^2)}{4}\right]_0^4 = \frac{\sin(16)}{4}.$$

Double Integrals in Polar Coordinates:

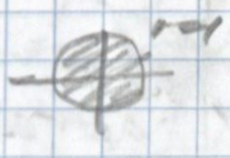


$r \geq 0$
$0 \leq \theta < 2\pi$

$x = r\cos\theta$
$y = r\sin\theta$

ex. Describe the following regions in polar coordinates

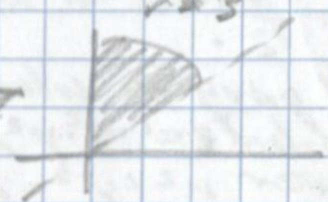
1. $D = \{(r,\theta) \mid r \leq 1\} = \{(r,\theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

unit disk $\Rightarrow$ 

2. $D = \{(r,\theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

half annulus $\Rightarrow$ 

3. $D = \{(r,\theta) \mid r \leq 3, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}\}$

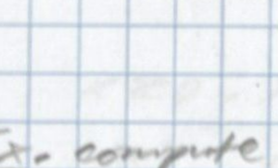
sector $\Rightarrow$ 

* $dA = r\,dr\,d\theta \Rightarrow D = \{(r,\theta) \mid r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$

$A = \frac{1}{2} r_2^2 \Delta\theta - \frac{1}{2} r_1^2 \Delta\theta = \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta$

$= r^* \Delta r \Delta\theta$ where $r^* = \frac{1}{2}(r_1 + r_2)$

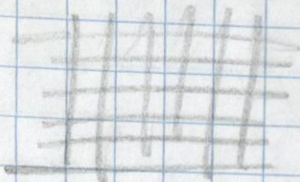
Ex. compute $\iint_D e^{x^2+y^2} dA$ where D is the unit disk



$$= \iint_{\substack{0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1}} e^{(r\cos\theta)^2 + (r\sin\theta)^2} r\,dr\,d\theta = \iint_{\substack{0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1}} e^{r^2} r\,dr\,d\theta$$

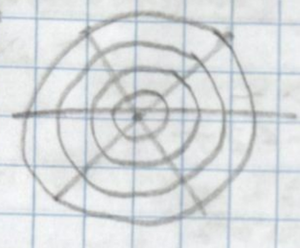
$\Rightarrow$ completely separable $\Rightarrow \left(\int_{r=0}^{r=1} e^{r^2} r\,d\theta\right)\left(\int_{\theta=0}^{\theta=2\pi} 1\,d\theta\right) = \left(\frac{e-1}{2}\right)(2\pi).$

[Lecture Notes] Polar Coordinates

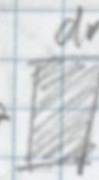


$dA = dx\,dy$

Cartesian grid



$\approx r\,d\theta$

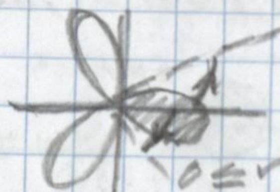


$A = r\,dr\,d\theta$

$* \; dA = r\,dr\,d\theta$

ex: Find the area enclosed by one petal of the rose

$r = \cos 3\theta$

$\hookrightarrow$  $\rightsquigarrow$ Area $= \iint 1\,dA$

$0 \le r \le \cos 3\theta$

$$\Rightarrow A = \int_{\theta=-\frac{\pi}{6}}^{\theta=\frac{\pi}{6}} \left(\int_{r=0}^{r=\cos 3\theta} r\,dr \right) d\theta$$

$\cos(3\theta) = 0$ when $3\theta = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

i.e. $\theta = \ldots, \underbrace{-\frac{\pi}{6}, \frac{\pi}{6}}, \frac{3\pi}{6}, \ldots$

length of interval $= \frac{\pi}{3}$

$$\Rightarrow \int_{\theta=-\frac{\pi}{6}}^{\theta=\frac{\pi}{6}} \left[\frac{r^2}{2} \right]_0^{\cos 3\theta} d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos^2 3\theta}{2}\,d\theta$$

$\cos 2x = 2\cos^2 x - 1$

$$\Rightarrow \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{4}\,d\theta = \ldots = \frac{\pi}{12}.$$

ex: Set up an integral giving the volume of the region bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the xy-plane, and inside the cylinder $x^2 + y^2 = 2y$

polar coordinates: paraboloid: cylinder: (inside: $\le$)

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$z = 4 - r^2$ $\qquad r^2 = 2r\sin\theta \Leftrightarrow r = 2\sin\theta$

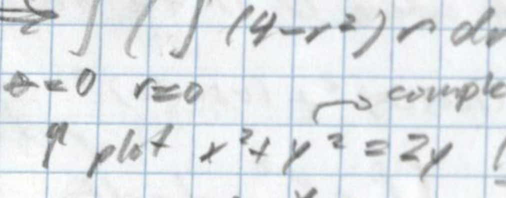
$$\text{Volume} = \iint_{\substack{\text{inside}\\\text{cylinder}}} z\,dA = \iint_{\substack{\text{inside}\\\text{cylinder}}} (4-r^2)\,r\,dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{\theta=\pi} \left(\int_{r=0}^{r=2\sin\theta} (4-r^2)\,r\,dr \right) d\theta$$

$\hookrightarrow$ complete square: $x^2 + y^2 - 2y + 1 = 0 + 1$

$x^2 + (y-1)^2 = 1$

¶ plot $x^2 + y^2 = 2y$ ($r = 2\sin\theta$) in plane



also check $z = 4 - r^2$ is ≥ 0 on this region ✓

Triple Integrals:

* $\iiint_D f(x,y,z)\,dV = \text{volume}(D)\cdot\begin{pmatrix}\text{average of}\\ f \text{ on } D\end{pmatrix}$

  dV = volume $(dx\,dy\,dz)$

* $\text{Volume}(D) = \iiint_D 1\,dV$

ex: Compute the triple integral of $f(x,y,z) = x^2ye^{xyz}$ over the box $B = \underbrace{[0,1]}_{x} \times \underbrace{[1,2]}_{y} \times \underbrace{[2,3]}_{z}$.

$$\iiint_B x^2ye^{xyz}\,dV = \int_{x=0}^{x=1}\int_{y=1}^{y=2}\int_{z=2}^{z=3} x^2ye^{xyz}\,dz\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{x=1}\int_{y=1}^{y=2}\left[xe^{xyz}\right]_{z=2}^{z=3} dy\,dx = \int_{x=0}^{1}\int_{y=1}^{2}\left(xe^{3xy} - xe^{2xy}\right)dy\,dx$$

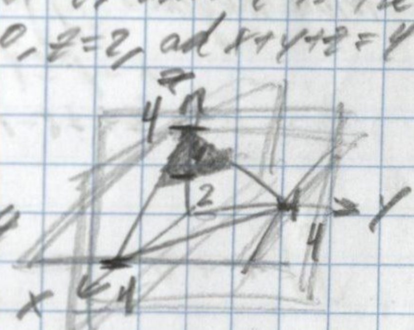
$$\Rightarrow \int_{x=0}^{x=1}\left[\frac{1}{3}e^{3xy} - \frac{1}{2}e^{2xy}\right]_{y=1}^{y=2} dx = \int_0^1 \left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x}\right) - \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)dx$$

$$\Rightarrow \left[\frac{e^{6x}}{18} - \frac{e^{4x}}{8} - \frac{e^{3x}}{9} + \frac{e^{2x}}{4}\right]_{x=0}^{x=1} = \ldots$$

* For region E, bounds look as follows:
  - x: "back to front"
  - y: "left to right"
  - z: "bottom to top"

ex: Set up the integral to compute the volume of E, where E is the tetrahedron bounded by the planes $x=0$, $y=0$, $z=2$, and $x+y+z=4$

$\text{volume}(E) = \iiint_E 1\,dV$



$x \geq 0,\ y \geq 0,\ z \geq 2,\ x+y+z \leq 4$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=2}^{z=4-y-x} 1\,dz\,dy\,dx$$

$z = 4 - x = z$

$y \leq 4 - x - 2$

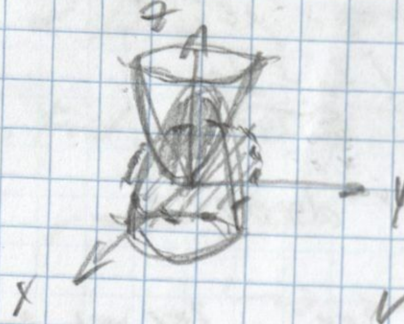
$x \leq 4 - y - z$

$\rightsquigarrow x \leq 4 - 0 - 2.$

Lecture Notes

ex. Polar coordinates problem: Find the volume bounded by the paraboloids $z=8-x^2-y^2$ and $z=x^2+y^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \qquad \Rightarrow z = 8-r^2 \qquad \Rightarrow z = r^2$$



upper: $z=8-r^2$
lower: $z=r^2$

Intersection in xy-plane:
$8-r^2=r^2$
$8=2r^2$
$4=r^2$
$2=r$

$$V = \iint_{r\le 2} (8-r^2)-r^2 \, dA$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (8-2r^2)\, r \, dr\, d\theta$$

$$= \left(\int_{\theta=0}^{\theta=2\pi} 1 \, d\theta\right)\left(\int_{r=0}^{r=2} 8r-2r^3 \, dr\right)$$

$$= 2\pi \cdot \left[4r^2 - \frac{r^4}{2}\right]_{r=0}^{r=2} = 2\pi\left[16-\frac{16}{2}\right] = 16\pi$$

triple integral: $\displaystyle\iiint_{\substack{x^2+y^2\le 4 \\ r^2\le z\le 8-r^2}} 1 \, dV$

$$\iint r+\theta = \iint r\cdot 1 + \iint \theta \cdot 1$$

ex: $\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz \, dz \, dx \, dy$ using $dy\,dz\,dx$.

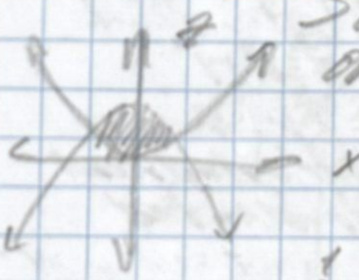
$0 \le y \le 4$ } y is ind of x, z

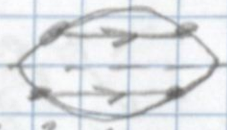
$-1 \le x \le 1$

$x^2 \le z \le 2-x^2$ } x, z depends on each other

$$= \int_{x=-1}^{x=1} \int_{z=x^2}^{z=2-x^2} \int_{y=}^{y=4} xyz \, dy \, dz \, dx$$

still in same order so can just rewrite





ex: Rewrite $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z) \, dz \, dy \, dx$ using $dx\,dy\,dz$

$0 \le x \le 1$

$0 \le y \le 1-x^2$

$0 \le z \le 1-x$

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=1} \int_{x=0}^{x=\min(\sqrt{1-y},\, 1-z)} f(x,y,z) \, dx \, dy \, dz$$

z: $z \ge 0$, $z \le 1-x$, largest when $x=0 \rightsquigarrow z \le 1$

y: $y \ge 0$, $y \le 1-x^2$, largest when $x=0 \rightsquigarrow y \le 1$

x: $0 \le x \le 1$, $y \le 1-x^2$, $z = 1-x$

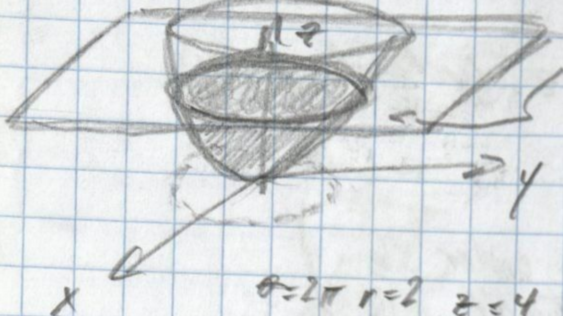
$x \ge 0$, $x \le 1$, $x \le \sqrt{1-y}$, $x \le 1-z$

* outside integral cannot depend on any other variables

* middle integral cannot depend on inside variables

Triple Integrals Using Cylinderical Coordinates:

ex: compute the volume of the solid bounded by $z=x^2+y^2$ and $z=4$.



region: $x^2+y^2 \le z \le 4$

volume: $\displaystyle\iiint_{x^2+y^2\le z\le 4} 1\,dV = \iint_{x^2+y^2\le 4}\left(\int_{z=x^2+y^2}^{z=4} 1\,dz\right)dA$

$\displaystyle\Rightarrow \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}\int_{z=x^2+y^2}^{z=4} 1\,dz\,(r\,dr\,d\theta)$ polar/cylinderical coordinates

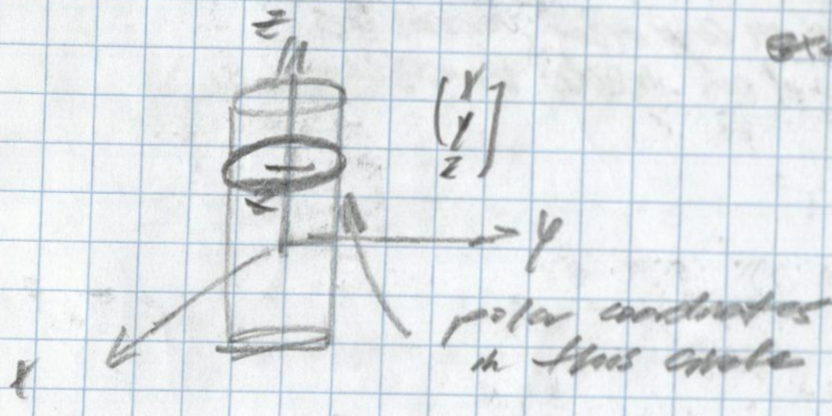
$(z=r^2)$

$\displaystyle\Rightarrow \left(\int_{\theta=0}^{\theta=2\pi} 1\,d\theta\right)\cdot\left(\int_{r=0}^{r=2}\int_{z=r^2}^{z=4} r\,dz\,dr\right) = 2\pi\int_{r=0}^{r=2} r(4-r^2)\,dr$

$\displaystyle = 2\pi\left[2r^2-\frac{r^4}{4}\right]_{r=0}^{r=2} = 2\pi(8-4) = 8\pi.$

(r,θ,z) where $\underbrace{x=r\cos\theta,\ y=r\sin\theta}_{(x,y)\text{ in polar coordinates}},\ \underbrace{z=z}_{z\text{ doesn't change}}$

$dV = r\,dr\,d\theta\,dz = dx\,dy\,dz$



ex: $\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} r\cos\theta\\ r\sin\theta\\ z=z\end{bmatrix}$

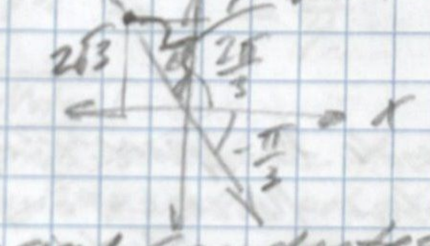
$\begin{bmatrix} 4\\ \pi/3\\ -2\end{bmatrix} = \begin{bmatrix} 2\\ 2\sqrt{3}\\ -2\end{bmatrix}.$

ex. Write $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2\sqrt{3} \\ 3 \end{bmatrix}$ in cylindrical coordinates

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}$ } polar coordinate in (x,y)

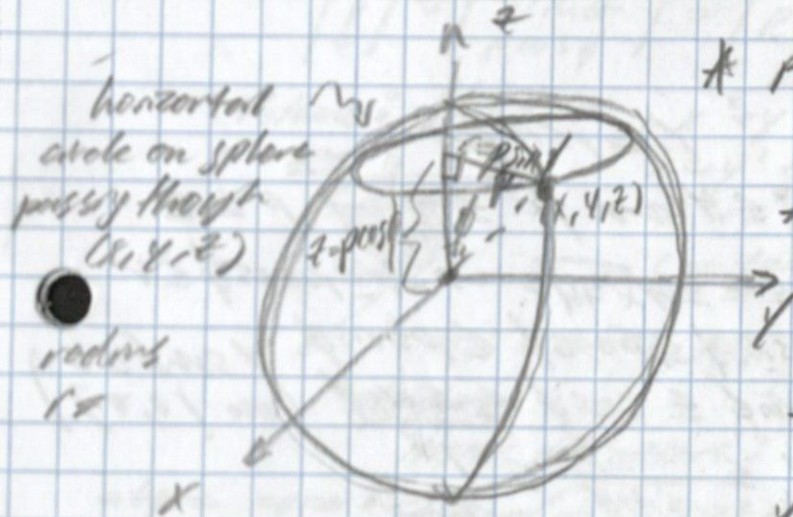
cylindrical coordinates: $(r, \theta, z) = (4, \frac{2\pi}{3}, 3)$

$\cancel{\frac{y}{x} = \tan\theta \rightsquigarrow \frac{2\sqrt{3}}{-2} = \tan\theta \rightsquigarrow -\sqrt{3} = \tan\theta \rightsquigarrow \theta} = \frac{2\pi}{3}$



$r = \sqrt{x^2+y^2}$
$= \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

Spherical Coordinates:

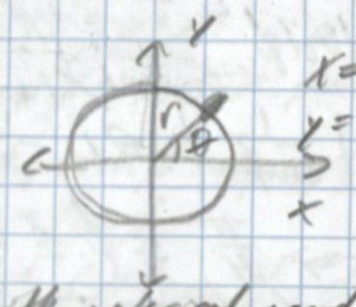


* ρ = distance from origin
$= \sqrt{x^2+y^2+z^2}$

* ϕ = angle from positive z-axis
(latitude measured from north pole)

* $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\rho\sin\phi)\cos\theta \\ (\rho\sin\phi)\sin\theta \\ \rho\cos\phi \end{bmatrix}$

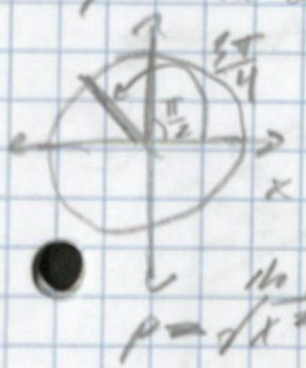
look at shadow of horizontal circle in xy-plane:



$x = r\cos\theta$
$y = r\sin\theta$

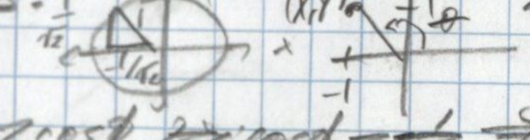
* (ρ, θ, ϕ)
$\rho \geq 0$
$0 \leq \theta \leq 2\pi$
$0 \leq \phi \leq \pi$

ex. write the point with spherical coordinates $(3, \frac{\pi}{2}, \frac{3\pi}{4}) \rightsquigarrow (\rho, \theta, \phi)$ in cartesian coordinates



$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3\cdot\sin\frac{3\pi}{4}\cdot\cos\frac{\pi}{2} \\ 3\cdot\sin\frac{3\pi}{4}\cdot\sin\frac{\pi}{2} \\ 3\cdot\cos\frac{3\pi}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3\sqrt{2}}{2} \\ -\frac{3\sqrt{2}}{2} \end{bmatrix}$

ex. write the point w/ cartesian coordinates $(-1, 1, -\sqrt{2})$ in spherical coordinates.



$\rho = \sqrt{x^2+y^2+z^2} = 2.$

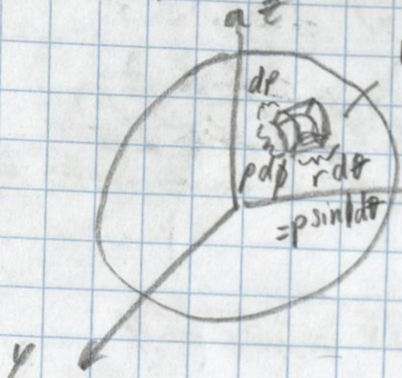
ϕ: use $z = \rho\cos\phi \rightsquigarrow -\sqrt{2} = 2\cos\phi \rightsquigarrow \cos\phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}.$

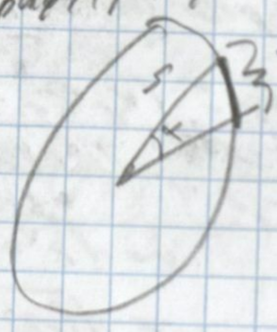
θ: (or use $x = \rho\sin\phi\cos\theta$ / $y = \rho\sin\phi\sin\theta$) $\tan\theta = \frac{y}{x} \Rightarrow \theta = \frac{3\pi}{4}.$

Change of Spherical Variables:

* $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$V \approx (d\rho)(\rho\, d\phi)(\rho \sin\phi \, d\theta)$

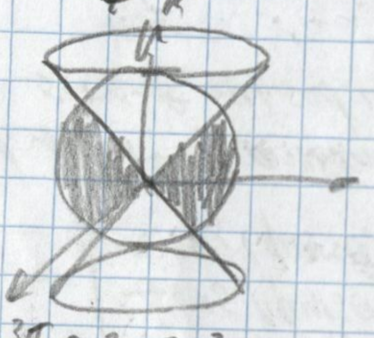




* Riemann sums of non-cartesian coordinates are deformed rectangles or rectangular prisms

ex Suppose R is the solid that lies inside the sphere $x^2+y^2+z^2=4$, under the cone $z=\sqrt{x^2+y^2}$ and above the cone $z=-\sqrt{x^2+y^2}$. Write the following triple integral in spherical coordinates

$$\iiint_R z^2 \, dV$$



$$\Rightarrow \int_{\phi=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} (\rho\cos\phi)^2 (\rho^2 \sin\phi) \, d\rho \, d\theta \, d\phi$$

algebra: ($r=\rho\sin\phi$)

$x^2+y^2+z^2 \le 4 \rightsquigarrow \rho^2 \le 4 \to 0 \le \rho \le 2$

$-\sqrt{x^2+y^2} \le z \le \sqrt{x^2+y^2} \rightsquigarrow -r \le \rho\cos\phi \le r$

$\rightsquigarrow -\rho\sin\phi \le \rho\cos\phi \le \rho\sin\phi$

$\rightsquigarrow -\sin\phi \le \cos\phi \le \sin\phi$ ($\sin\phi \ge 0$ on $[0,\pi]$)

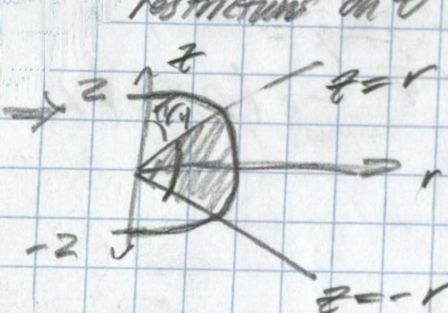
$\rightsquigarrow -1 \le \underbrace{\frac{\cos\phi}{\sin\phi}}_{1/\tan\phi} \le 1$

$\Rightarrow \left|\frac{1}{\tan\phi}\right| \le 1 \rightsquigarrow |\tan\phi| \ge 1$

$\Rightarrow \frac{\pi}{4} \le \phi \le \frac{5\pi}{4}$

no restrictions on θ ∴ $0 \le \theta \le 2\pi$

geometry:
restrictions on θ



$$\Rightarrow \int_{\phi=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} (\rho\cos\phi)^2 (\rho^2\sin\phi) \, d\rho \, d\theta \, d\phi$$

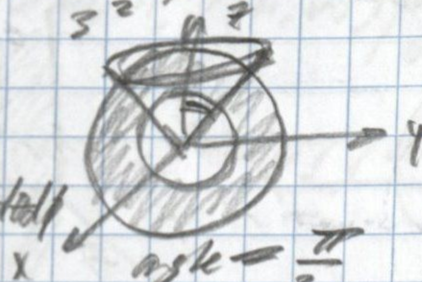
$= \dots = \frac{32\sqrt{2}\pi}{15}$.

$\le \rho \le 2$

$\le \phi \le \frac{3\pi}{4}$

ex. Suppose E is the region bounded by the spheres $x^2+y^2+z^2=4$, $x^2+y^2+z^2=9$, and above the cone $\phi = \frac{\pi}{3}$.

Evaluate: $\iiint_R \frac{4}{65} z \, dV$



$2 \le \rho \le 3$

$0 \le \phi \le \frac{\pi}{3}$

$0 \le \theta \le 2\pi$

angle $= \frac{\pi}{3}$

$$\Rightarrow \int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=0}^{2\pi} \int_{\rho=2}^{\rho=3} \left(\frac{4}{65}\rho\cos\phi\right)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi = \dots = \frac{2\pi}{4}$$

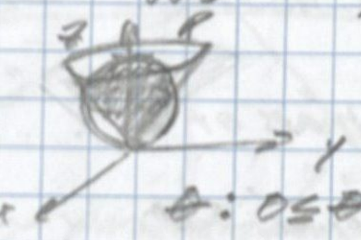
Lecture Notes

10.23.23

ex. find the volume of the region inside the sphere $x^2+y^2+z^2=4z$ and above the cone $z=\sqrt{\frac{1}{3}(x^2+y^2)}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \rho\cos\phi \end{bmatrix} = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix}; \quad \begin{matrix} \rho \ge 0 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}$$

Volume: $\iiint 1 \, dV$; $x^2+y^2+z^2-4z=0$ $(r=\rho\sin\phi)$



$x^2+y^2+z^2-4z+4=4$

$x^2+y^2+(z-2)^2=2^2$

center: $(0,0,2)$, radius $=2$

θ: $0 \le \theta \le 2\pi$ (rotationally symmetric about z-axis)

algebra:

inside the sphere: $x^2+y^2+z^2 \le 4z \Rightarrow \rho^2 \le 4\rho\cos\phi$

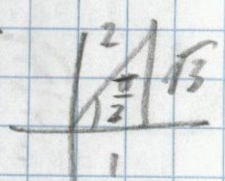
$\Rightarrow \rho \le 4\cos\phi$

(also $\phi \le \frac{\pi}{2}$)

above cone: $z=\sqrt{\frac{1}{3}(x^2+y^2)}$

$\Rightarrow \rho\cos\phi \ge \sqrt{\frac{1}{3}r^2} \Rightarrow \rho\cos\phi \ge \sqrt{\frac{1}{3}}\,r$

$\Rightarrow \rho\cos\phi \ge \frac{1}{\sqrt{3}}\rho\sin\phi \Rightarrow \sqrt{3} \ge \tan\phi \Rightarrow \phi \le \frac{\pi}{3}$

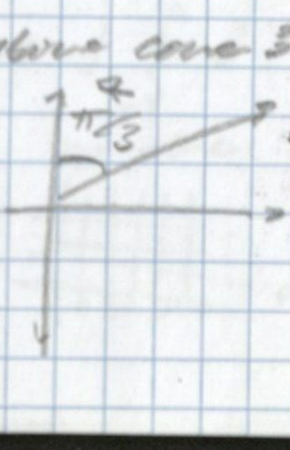


$$\text{Volume} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{3}} \int_{\rho=0}^{\rho=4\cos\phi} 1\,\underbrace{(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta}_{dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta} = \dots = 10\pi.$$

geometric: inside sphere:

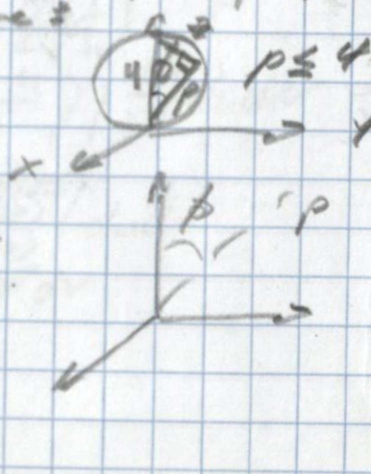
$\rho \le 4\cos\phi$

above cone:



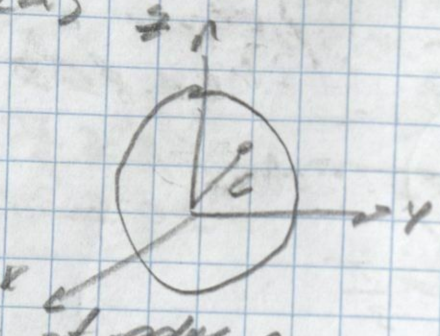
$z=\sqrt{\frac{1}{3}}\,r^2$

slope $=\frac{1}{\sqrt{3}}$



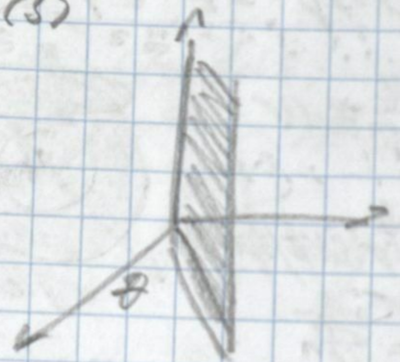
ex. Describe the surfaces whose equations in spherical coordinates are (a) $\rho = c$, (b) $\theta = c$, (c) $\phi = c$

(a)



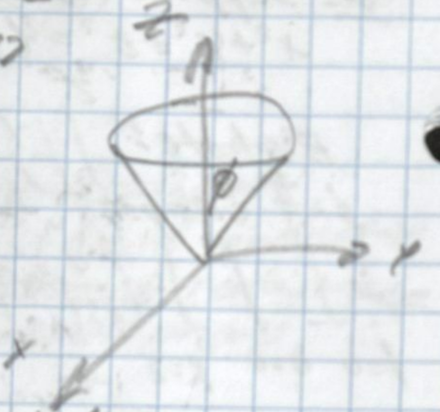
sphere of radius c (if $c \geq 0$)

(b)



half-plane in direction of θ in xy plane

(c)



half-cone

Mass and Center of Mass:

recall: $D \subseteq \mathbb{R}^2$: $\iint_D 1\, dA = \text{area}(D)$

$$\frac{\iint_D f(x,y)\,dA}{\iint_D 1\, dA} = \text{average of } f \text{ on } D$$

* density $\rho(x,y)$ on D: $\iint_D \rho(x,y)\,dA = \text{mass of } D$

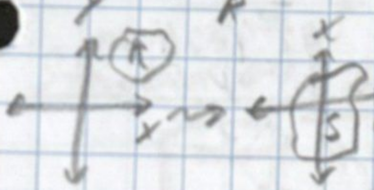
$$\frac{\iint_D f(x,y)\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} = \text{average of } f \text{ on } D \text{ weighted with density}$$

* Center of mass of D: average x-value w/ density ρ, average y-value w/ density ρ

$$(\bar{x}, \bar{y}) = \left(\frac{\iint_D x\rho\,dA}{\iint_D \rho\,dA}, \frac{\iint_D y\rho\,dA}{\iint_D \rho\,dA} \right)$$

Change of Variables:

$$* \iint_R f(x,y)\,dA = \iint_S f(x(u,v),\,y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

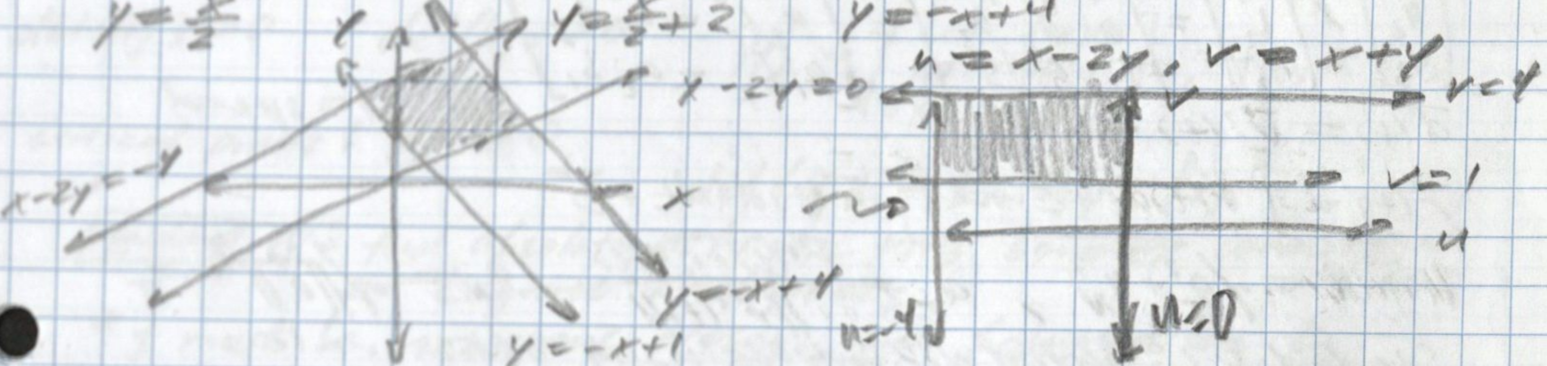


$$= \iint_{T^{-1}(R)} f(T(u,v))\,|\det DT|\,du\,dv$$

* Jacobian Matrix:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

ex. Compute $\iint_D 3xy\,dA$ where D is the region bounded by $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.



* every parallelogram becomes a rectangle under the right change of variables! *

$$\iint_D 3xy\,dA = \int\int 3\Big(\underbrace{\frac{u+2v}{3}}_{x \text{ in terms of } u,v}\Big)\Big(\underbrace{\frac{v-u}{3}}_{y \text{ in terms of } u,v}\Big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

① $x - 2y = u$

② $x + y = v$

① + 2② $\Rightarrow 3x = u + 2v$

② − ① $\Rightarrow 3y = v - u$

$\therefore\ x = \frac{u+2v}{3},\ y = \frac{v-u}{3}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) - \left(\tfrac{2}{3}\right)\left(-\tfrac{1}{3}\right) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

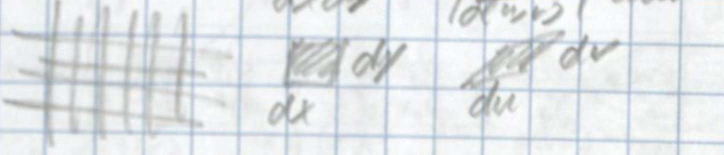
$$\Rightarrow \int_1^4 \int_{-4}^0 3\left(\frac{u+2v}{3}\right)\left(\frac{v-u}{3}\right)\left|\frac{1}{3}\right| du\,dv$$

or use $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (-2)(1) = 3$$

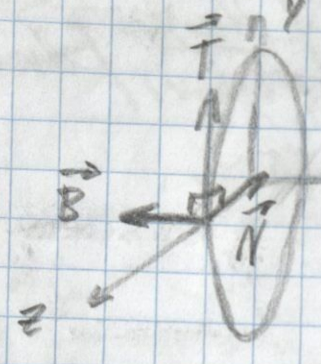
xy-plane area: $dx\,dy$

area: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$



[Independent Notes] Exam 2: Standards 06-11 10.25.23

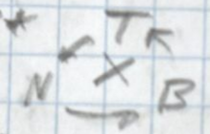
Standard 06: TNB frame, planes, and motion in space



$$\vec T = \frac{\vec r'(t)}{\|\vec r'(t)\|} \qquad \vec N = \frac{\vec T'(t)}{\|\vec T'(t)\|}$$

$$\vec B = \frac{\vec r'(t) \times \vec r''(t)}{\|\vec r'(t) \times \vec r''(t)\|}$$

3 orthogonal unit-vectors



$$\begin{pmatrix} \vec T = \vec N \times \vec B \\ \vec N = \vec B \times \vec T \\ \vec B = \vec T \times \vec N \end{pmatrix}$$

normal plane: $\perp \vec T = \vec n$

osculating plane: $\perp \vec B = \vec n$

$ax + by + cz = d \longrightarrow T, N, B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \vec r(0) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

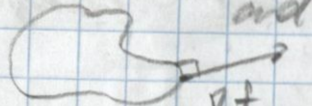
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{vmatrix} \hat i & \hat j & \hat k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$\vec a(t) = \vec v'(t) = \vec r''(t)$ $\qquad \|\vec v(t)\| = $ speed

$\vec r(t) = \int v(t)\,dt + \vec C = \iint \vec a(t)\,dt\,dt + \vec D$

$\|\vec a\| = \sqrt{a_T^2 + a_N^2}, \quad a_T = \frac{\vec r'(t) \cdot \vec r''(t)}{\|\vec r'(t)\|}, \quad a_N = \sqrt{\|\vec a\|^2 - a_T^2}$

Standard 07: partial derivatives using chain rule and implicit differentiation and gradient



$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \lim_{h \to 0} \begin{bmatrix} \frac{f(x+h,y,z) - f(x,y,z)}{h} \\ \frac{f(x,y+h,z) - f(x,y,z)}{h} \\ \frac{f(x,y,z+h) - f(x,y,z)}{h} \end{bmatrix}$$

$\nabla f \perp f = c$

$$\frac{\partial}{\partial x} fg = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i} \longrightarrow$$

$\frac{\partial z}{\partial x}$ $\quad z \quad$ $\frac{\partial z}{\partial y}$

$x \qquad\qquad y$

$\frac{\partial x}{\partial s} \quad \frac{\partial x}{\partial t} \quad \frac{\partial y}{\partial s} \quad \frac{\partial y}{\partial t}$

$s \quad t \quad s \quad t$

$\vdots$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

ex: $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$.

ex: when computing $\frac{\partial x}{\partial z}$ implicitly, $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial x}{\partial y} = 0$.

Standard 08: Directional derivatives and extrema of ROC

$D_{\vec{u}} f = \nabla f \cdot \hat{u} = \|\nabla f\| \|\hat{u}\| \cos\theta = \|\nabla f\| \cos\theta$

max ROC of $f = \|\nabla f\|$ in direction ∇f $\quad \theta = 0°$

min ROC of $f = -\|\nabla f\|$ in direction $-\nabla f$

Standard 09: find local extrema

paraboloid up	paraboloid down	saddle	plane
local min	local max	neither	inconclusive
$f_{xx} > 0, f_{yy} > 0$	$f_{xx} < 0, f_{yy} < 0$	$\det(H_f) < 0$	$\det(H_f) = 0$
$\det(H_f) > 0$	$\det(H_f) > 0$		

$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$; $f_{xy} = f_{yx}$

critical points: $\nabla f = 0$

$\det(H_f) = f_{xx} f_{yy} - f_{xy}^2$

Standard 10: find absolute extrema using boundary conditions or Lagrange Multipliers

* f must be continuous, closed, and bounded on D

1. critical points $\nabla f = 0$
2. parameterize boundary $\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} a \\ b \end{bmatrix}$, $L \le t \le U$

a) $g_n(t) = t f(a, b) = m$

b) $g_n'(t) = 0$, $g(L)$, $g(U)$

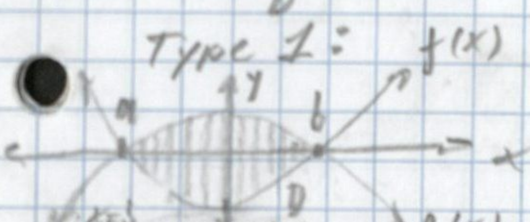
$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = c \\ h = k \end{cases}$ solve for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow$ plug in points, discern extrema

Standard 11: double integrals over simple regions

$V = \lim_{m,n \to \infty} \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i, y_j) \Delta x \Delta y = \iint f(x,y)\, dA$

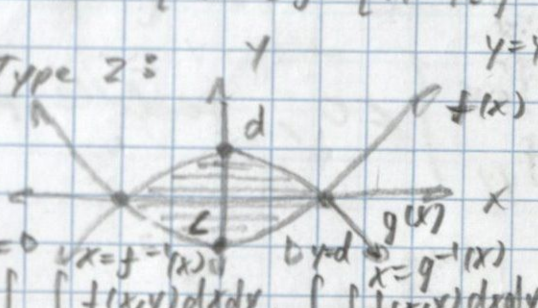
$\Rightarrow \iint_D f(x,y)\, dA$ when $D = [x_1, x_2] \times [y_1, y_2] = \int_{y=y_1}^{y=y_2} \int_{x=x_1}^{x=x_2} f(x,y)\, dx\, dy$

Type 1:

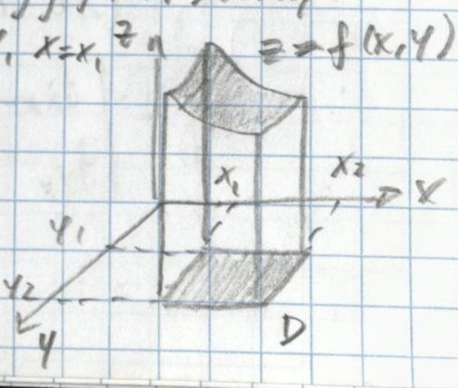


$= \int_{x=a}^{x=b} \int_{y=f(x)}^{y=g(x)} f(x,y)\, dy\, dx$

Type 2:



$= \int_{y=c}^{y=0} \int_{x=-f^{-1}(x)}^{x=f^{-1}(x)} f(x,y)\, dx\, dy + \int_{y=0}^{y=d} \int_{x=-g^{-1}(x)}^{x=g^{-1}(x)} f(x,y)\, dx\, dy$



Lecture Notes Exam 2 Review 10.25.23

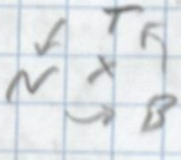
1. Find $\vec{T}$, $\vec{N}$, and $\vec{B}$ for the curve $\vec{r}(t) = \begin{bmatrix} 2\sin t \\ t \\ 2\cos t \end{bmatrix}$

$$\vec{r}\,'(t) = \begin{bmatrix} 2\cos t \\ 1 \\ -2\sin t \end{bmatrix}, \quad \vec{r}\,''(t) = \begin{bmatrix} -2\sin t \\ 0 \\ -2\cos t \end{bmatrix}$$

$$\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{(2\cos t)^2 + 1^2 + (-2\sin t)^2}}\,\vec{r}\,'(t) = \frac{1}{\sqrt{5}} \begin{bmatrix} 2\cos t \\ 1 \\ -2\sin t \end{bmatrix}$$

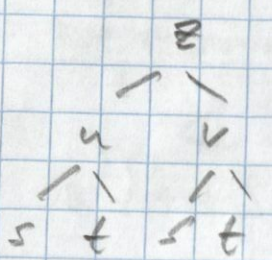
$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{bmatrix} -2\cos t \\ 4\sin^2 t + 4\cos^2 t \\ -2\sin t \end{bmatrix} = \begin{bmatrix} -2\cos t \\ 4 \\ -2\sin t \end{bmatrix}$$

$$\vec{B} = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|} = \frac{1}{\sqrt{20}} \begin{bmatrix} -2\cos t \\ 4 \\ -2\sin t \end{bmatrix}$$



$$\vec{N} = \vec{B} \times \vec{T} =$$

2. Find $\frac{\partial z}{\partial s}$, where $z = \tan\left(\frac{u}{v}\right)$ and $u = 2s + 3t$, $v = 3s - 2t$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{du}{ds} + \frac{\partial z}{\partial v}\frac{dv}{ds}$$

$$= \sec^2\left(\frac{u}{v}\right) \cdot \frac{2}{v} - \frac{1}{v^2}\sec^2\left(\frac{u}{v}\right) \cdot 3u$$

$$= \frac{2}{v}\sec^2\left(\frac{u}{v}\right) - \frac{3u}{v^2}\sec^2\left(\frac{u}{v}\right)$$

3. Consider $f(x,y) = \sin(2xy)$. At the point $(1,0)$, find the maximum rate of change of f and its direction

$$\nabla f = \begin{bmatrix} 2y\cos(2xy) \\ 2x\cos(2xy) \end{bmatrix} \overset{\text{at } (1,0)}{=} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \text{direction}$$

$$\|\nabla f\| = 2 \Rightarrow \text{magnitude}$$

4. consider $f(x,y) = e^{xy}$

a) find and classify all critical points of f

$$\nabla f = \begin{bmatrix} ye^{xy} \\ xe^{xy} \end{bmatrix} = 0 \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} < 0, \text{ saddle point}$$

5) find the maximum value of f subject to the constraint $x^3+y^3=16$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=16 \end{cases} \to \begin{bmatrix} ye^{xy} \\ xe^{xy} \end{bmatrix} = \lambda \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} ye^{xy} = 3\lambda x^2 \\ xe^{xy} = 3\lambda y^2 \\ x^3+y^3=16 \end{cases} \Rightarrow \begin{cases} e^{xy} = \frac{3\lambda x^2}{y} \\ e^{xy} = \frac{3\lambda y^2}{x} \end{cases} \Rightarrow \frac{3\lambda x^2}{y} = \frac{3\lambda y^2}{x}$$

$$\Rightarrow 3\lambda x^3 = 3\lambda y^3$$

$$x^3 = y^3$$

$$\Rightarrow 2x^3 = 16$$

$$x^3 = 8$$

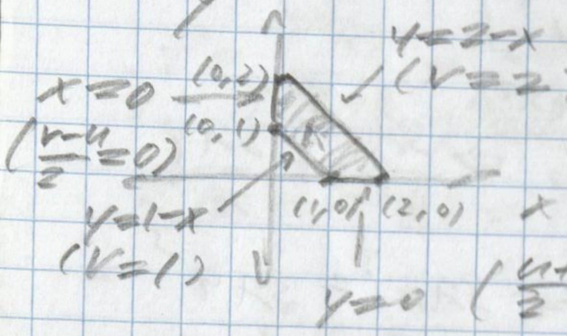
$$x = 2 \Rightarrow y = 2$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 1 \qquad \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow e^4 \Rightarrow \text{max}$$

[Lecture Notes]

10.27.23

ex. compute $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region of vertices $(1,0)$, $(2,0)$, $(0,2)$, $(0,1)$.

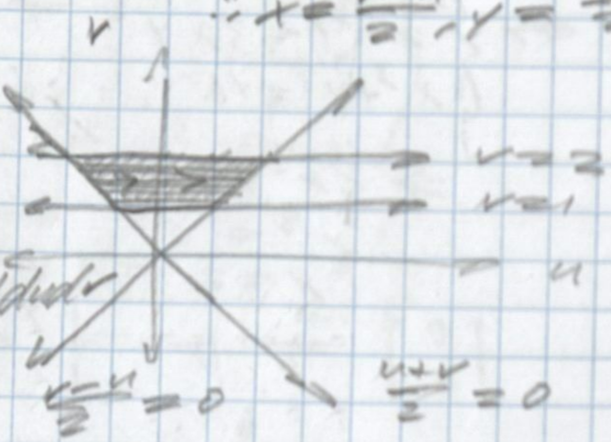


$u = y - x$ ① $y - x = u$ ①+② ⇒ $2y = u + v$

$v = y + x$ ② $y + x = v$ ②−① ⇒ $2x = v - u$

$\therefore x = \frac{v-u}{2}$, $y = \frac{u+v}{2}$

⇒



$$\Rightarrow \iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \iint \cos\left(\frac{u}{v}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_{v=1}^{v=2}\int_{u=-v}^{u=v} \cos\left(\frac{u}{v}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(1) - (1)(1) = -2, \quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$= \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\left|-\frac{1}{2}\right| du\,dv = \ldots = \frac{3}{2}\sin(1).$$

3D Change of Variables:

* $\iiint_R f(x,y,z)\, dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

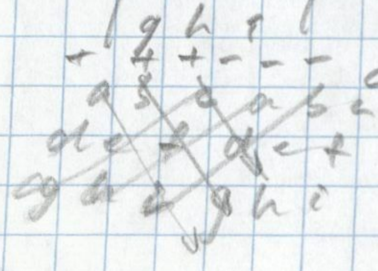
where $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$ # can be used in any n dimensions #

ex. let $x = \frac{u}{v}$ and $y = uv$. compute the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \left(\frac{1}{v}\right)(u) - \left(-\frac{u}{v^2}\right)(v)$$

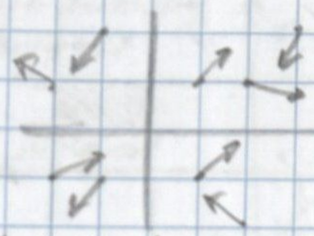
$$= \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}.$$

* $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{matrix} aei & -afh \\ + & - \\ bfg & bdi \\ + & - \\ cdh & ceg \end{matrix}$ or $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$



Vector fields:

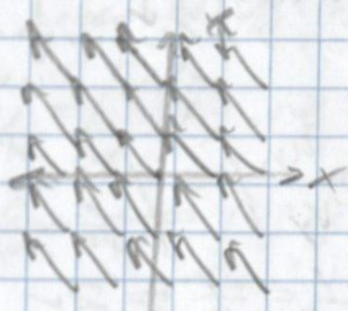
a vector field in $\mathbb{R}^2$ is a function $\vec{F}$ which is assigned to each point (x,y) in its domain a 2D vector $\vec{F}(x,y)$. possible in any other dimension

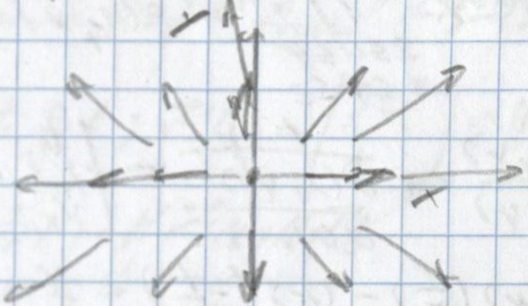


* $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$
  $= \langle P, Q \rangle$
* $\vec{F}(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$

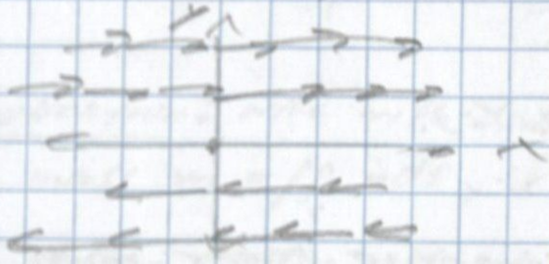
ex. Sketch the vector fields

$\vec{F}(x,y) = \langle -1, 1 \rangle$, $\vec{F}(x,y) = (x,y)$





$\vec{F}(x,y) = \langle 2y, 0 \rangle$



constant on horizontal line

|Lecture Notes|

10.31.23

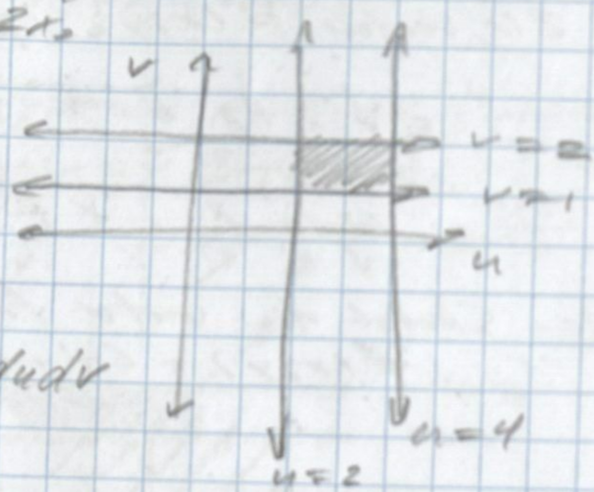
ex: Evaluate $\iint_D xy\,dx\,dy$, where D is the region in the first quadrant bound by the curves $xy=2$, $xy=4$, $y=x$, and $y=2x$.

$xy=2 \Rightarrow u=2$

$xy=4 \Rightarrow u=4$

$y=x \Rightarrow \frac{y}{x}=1 \Rightarrow v=1$

$y=2x \Rightarrow \frac{y}{x}=2 \Rightarrow v=2$



$$\Rightarrow \iint_D xy\,dx\,dy = \iint_{\substack{2\le u\le 4\\ 1\le v\le 2}} u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= (y)\left(\frac{1}{x}\right) - (x)\left(-\frac{y}{x^2}\right) = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x} = 2v$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{|2v|} = \frac{1}{2v}$$

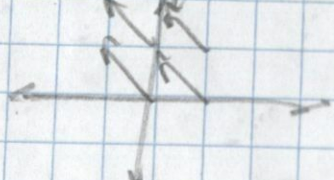
$$\Rightarrow \int_{v=1}^{v=2}\int_{u=2}^{u=4} \frac{u}{2v}\,du\,dv = \ldots = 3\ln(2).$$

Gradient Vector Field: (conservative vector field)

* vector field of form ∇f for some function $\underline{f(x,y) \text{ or } f(x,y,z)}$ potential

* $\nabla f(x,y) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ & $\nabla f(x,y,z) = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$

ex. $\vec{F} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the gradient vector field when $f = y - x$



$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Line Integral

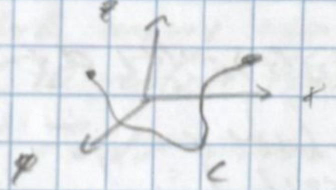
* $\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt \leftarrow \left(\begin{array}{c}\text{average of}\\ f \text{ on } C\end{array}\right)\cdot\left(\begin{array}{c}\text{length}\\ \text{of}\\ C\end{array}\right)$

single variable integral: $\int_a^b f(x)\,dx$ $\longleftarrow a \quad b \longrightarrow \mathbb{R}$

ds = arc length

(i.e. unit speed parametrization)

line integral



$\vec{r}(t)$ is parametrization of C

ex. compute the line integral of $f(x,y) = 18y^3$ along the piece of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$

$$C: \vec{r}(t) = \begin{bmatrix} t^3 \\ t \end{bmatrix};\ -1 \le t \le 1;\ \vec{r}\,'(t) = \begin{bmatrix} 3t^2 \\ 1 \end{bmatrix};\ \|\vec{r}\,'(t)\| = \sqrt{(3t^2)^2+1} = \sqrt{9t^4+1}$$

$$\text{line integral} = \int_{-1}^{1} f(t^3, t)\cdot\|\vec{r}\,'(t)\|\,dt$$

$$= \int_{-1}^{1} \overbrace{(18t^3)\underbrace{\sqrt{9t^4+1}}_{\text{sub } u = 9t^4+1}}^{\text{odd function}}\,dt = \ldots = 0.$$

application: wire with linear density ρ

* mass $m = \int_C \rho\,ds$, $\bar{x} = \frac{1}{m}\int_C x\rho\,ds$, $\bar{y} = \frac{1}{m}\int_C y\rho\,ds$, $\bar{z} = \frac{1}{m}\int_C z\rho$

other types of line integrals:

* replace Δs to Δx_i or Δy_i

* x: $\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$ projecting onto x-axis

* y: $\int_C f(x,y)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt$ projecting onto y-axis

case when C is parameterized by $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$

* $\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P\,dx + \int_C Q\,dy$

ex: compute the line integral $\int_C y\,dx - x\,dy + z\,dz$ where C is the path given by:

a) C_1 : $\vec{r}_1(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$

b) C_2 : line segment from $(1,0,0)$ to $(1,0,2\pi)$

c) C_3 : line segment from $(1,0,2\pi)$ to $(1,0,0)$

$$= \int_a^b y(t)x'(t) - x(t)y'(t) + z(t)z'(t)\,dt$$

(a) $\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$, $\vec{r}\,' = \begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$

$$\Rightarrow \int_0^{2\pi} (\sin t)(-\sin t) - (\cos t)(\cos t) + (t)(1)\,dt$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t + t)\,dt = \int_0^{2\pi} (-1+t)\,dt = \left[-t + \frac{t^2}{2}\right]_0^{2\pi} = -2\pi + 2\pi^2.$$

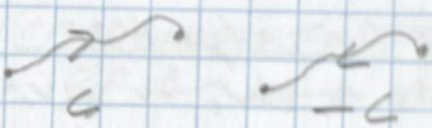
(b) $\begin{bmatrix} 0 \\ 0 \\ 2\pi \end{bmatrix}$

$(1,0,0)$ $(1,0,2\pi)$ $\vec{r}(t) = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{initial point}} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2\pi \end{bmatrix}}_{\text{direction}} = \begin{bmatrix} 1 \\ 0 \\ 2\pi t \end{bmatrix}$; $0 \le t \le 1$

$$\vec{r}\,'(t) = \begin{bmatrix} 0 \\ 0 \\ 2\pi \end{bmatrix}$$

$$\Rightarrow \int_0^1 (0)(0) - (1)(0) + (2\pi t)(2\pi)\,dt = \int_0^1 4\pi^2 t\,dt = 2\pi^2.$$

(c) same curve as (b), with opposite orientation.



$\therefore -2\pi^2$

line integral reflections:

* $\int_{-C} f\,dx = -\int_C f\,dx$
* $\int_{-C} f\,dy = -\int_C f\,dy$
* $\int_{-C} f\,dz = -\int_C f\,dz$
* $\int_{-C} f\,ds = \int_C f\,ds$ $\therefore$ orientation doesn't matter for arc length line integral

Work:

unit tangent vector

$$* \; W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\left(\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right)\|\vec{r}\,'(t)\|\,dt$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

"Work moving along C in vector field $\vec{F}$" $\quad \vec{F} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot d\vec{r} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$

Ex: compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \begin{bmatrix} xy \\ 3y^2 \end{bmatrix}$ and C is parametrized by $\vec{r}(t) = \begin{bmatrix} 11t^4 \\ t^3 \end{bmatrix}$; $0 \le t \le 1$

$$= \int_0^1 \vec{F}(\underbrace{11t^4}_{x}, \underbrace{t^3}_{y}) \cdot \vec{r}\,'\,dt$$

$$= \int_0^1 \begin{bmatrix} (11t^4)(t^3) \\ 3(t^3)^2 \end{bmatrix} \cdot \begin{bmatrix} 44t^3 \\ 3t^2 \end{bmatrix} dt$$

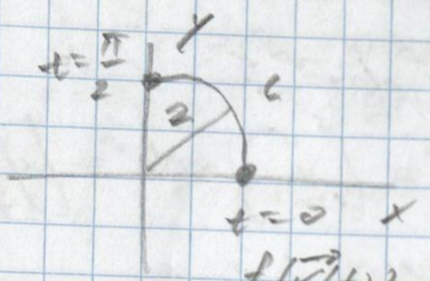
dot product

$$= \int_0^1 (11t^7)(44t^3) + (3t^6)(3t^2)\,dt = \ldots = 44+1.$$

Lecture Notes

11.1.23

ex. Find $\int_C xy\,ds$, where C is the portion of the circle $x^2+y^2=4$ in the first quadrant



$$\vec{r}(t)=\begin{bmatrix}2\cos t\\2\sin t\end{bmatrix},\quad 0\le t\le\frac{\pi}{2}$$

$$\vec{r}\,'(t)=\begin{bmatrix}-2\sin t\\2\cos t\end{bmatrix}\Rightarrow\|\vec{r}\,'(t)\|=2$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\underbrace{x(t)y(t)}_{f(\vec{r}(t))}\|\vec{r}\,'(t)\|\,dt=\int_0^{\frac{\pi}{2}}(2\cos t)(2\sin t)(2)\,dt$$

$$=\int_0^{\frac{\pi}{2}}8\cos t\sin t\,dt=\left[4\sin^2 t\right]_0^{\frac{\pi}{2}}=4\sin^2\tfrac{\pi}{2}-4\sin^2 0=4.$$

$u=\sin t$

or $\sin(2t)=2\sin t\cos t$

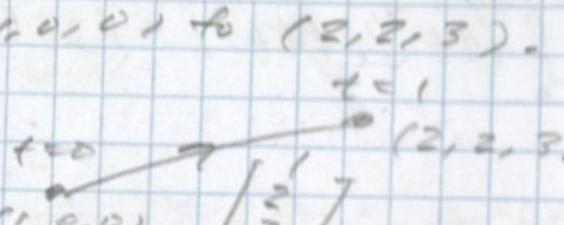
Sum of 3 line integrals w/respect to x, y, z

← Sum of work integral w/respect to vector field

ex. Find $\int_C z\,dx+xy\,dy+y^2\,dz$, where C is the line segment from $(1,0,0)$ to $(2,2,3)$.

$$\vec{r}(t)=\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}1\\2\\3\end{bmatrix},\quad 0\le t\le 1$$

initial point direction $\begin{pmatrix}t=0\text{ to}\\t=1\end{pmatrix}$



$$\vec{r}(t)=\begin{array}{c}\begin{bmatrix}1+t\\2t\\3t\end{bmatrix}\end{array}\begin{array}{c}x\\y\\z\end{array},\quad\vec{r}\,'(t)=\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{array}{c}x'\\y'\\z'\end{array}$$

$$=\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)$$

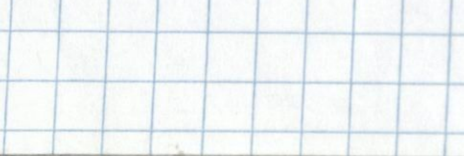
$$=\begin{bmatrix}3t\\(1+t)(2t)\\(2t)^2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\Rightarrow\int_0^1\left[z(t)x'(t)+x(t)y(t)y'(t)+y(t)^2z'(t)\right]dt$$

$$=\int_0^1\left[(3t)(1)+(1+t)(2t)(2)+(2t^2)(3)\right]dt=\ldots=\frac{3}{2}+\frac{10}{3}+4.$$

* can also write line integral $\int_C\vec{F}\cdot d\vec{r}$ where $\vec{F}=\begin{bmatrix}z\\xy\\y^2\end{bmatrix}$

* $W=\int_C\vec{F}\cdot\vec{T}\,ds=\int_a^b\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$



ex. Find the work done by $\vec{F} = \langle e^{z}, -xz, x+y \rangle$ in moving a particle from the origin to $(1,1,-1)$ along $\vec{r}(t) = \langle t^2, t^3, -t \rangle$

$$\vec{r} = \begin{bmatrix} t^2 \\ t^3 \\ -t \end{bmatrix},\ 0 \le t \le 1;\ \vec{r}\,'(t) = \begin{bmatrix} 2t \\ 3t^2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^1 \begin{bmatrix} e^{-t} \\ (t^2)(-t) \\ t^2+t^3 \end{bmatrix} \cdot \begin{bmatrix} 2t \\ 3t^2 \\ -1 \end{bmatrix} dt$$

$$= \int_0^1 \left[(e^{-t})(2t) + (-t^3)(3t^2) + (t^2+t^3)(-1) \right] dt$$

$$= \ldots = (2 - 4e^{-1}) - \frac{1}{2} - \frac{7}{12}.$$

The Fundamental Theorem of Line Integrals:

$$\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a) \leftarrow \text{fundamental theorem of calculus}$$

(gradient $\Rightarrow \vec{F} = \nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$)

If $\vec{F} = \langle P, Q \rangle$ is conservative, then $P = f_x$ and $Q = f_y$ for some $f = f(x,y)$. By Clairaut's theorem, $P_y = Q_x$ (as long as $\vec{F}$ is C^1). This serves as a test for conservative vector fields. Let $\vec{F}$ be a C^1 vector field

a) if $\vec{F}(x,y) = \langle P, Q \rangle$ and $\vec{F}$ is conservative then $P_y = Q_x$

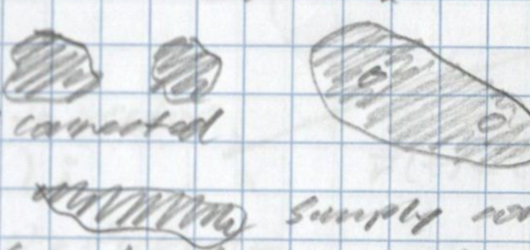
b) if $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ and $\vec{F}$ is conservative then

$$\text{curl}\,\vec{F} = \vec{0} \Leftrightarrow \begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$$

If $\vec{F} = \langle P, Q \rangle$ is a C^1 vector field on an open, $\underbrace{\text{simply connected}}$ region D, and $P_y = Q_x$, we have $\vec{F}$ is conservative

connected and any two points are connected by a curve i.e. no holes

(C^1 = has continuous partial derivatives)



not connected

connected but not simply connected

simply connected

If $\vec{F} = \langle P, Q, R \rangle$ is a C^1 vector field on all of $\mathbb{R}^3$ and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Is $\vec{F}$ conservative? If so find a potential function f.

(+C) is optional because only 1 function is needed

a) $\vec{F}(x,y) = \begin{bmatrix} x^2+y^2 \\ xy^2 \end{bmatrix}$ on $\mathbb{R}^2$ (simply connected)   (b) $\vec{F}(x,y) = \begin{bmatrix} ye^x \\ e^x+e^y \end{bmatrix}$ on $\mathbb{R}^2$

$\therefore$ conservative $\Leftrightarrow P_y = Q_x$

(a) $P_y \overset{?}{=} Q_x \Leftrightarrow 2y \overset{?}{=} y^2$ no! $\therefore$ not conservative / gradient

(b) $P_y \overset{?}{=} Q_x \Leftrightarrow e^x \overset{?}{=} e^x$ yes! now find f where $\nabla f = \vec{F}$.

i.e. $\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} ye^x \\ e^x+e^y \end{bmatrix} \to f_x = ye^x \Rightarrow f = \int ye^x\,dx = ye^x + g(y)$

plug into $f_y = Q$

$f_y = e^x + e^y \Rightarrow e^x + g'(y) = e^x + e^y \Rightarrow g'(y) = e^y \Rightarrow g(y) = \int e^y\,dy = e^y (+C)$

$\therefore f = ye^x + g(y) = ye^x + e^y (+C)$

strategy to find $f(x,y)$: (1) $f = \int P\,dx = h(x) + g(y)$ "constant of integration"
(2) plug into $f_y = Q$ to solve for $g(y)$

The Fundamental Theorem of Line Integrals:

* $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

* no theorem if our vector field is <u>not</u> conservative

$\nabla f = \langle f_x, f_y, f_z \rangle$

$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$

$= f(\vec{r}(b)) - f(\vec{r}(a))$

Tutorial Notes

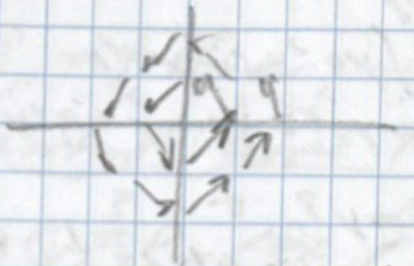
11.2.23

Line Integrals : $f: \mathbb{R}^3 \to \mathbb{R}$

Vector Fields : $F: \mathbb{R}^3 \to \mathbb{R}^3$

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$F(x,y) = -y\hat{i} + x\hat{j}$



Line Integrals (w/ respect to arc length):

$\int_C xy^4\,ds$ $\quad C$ = right half of $x^2+y^2=16$ (counterclockwise)

$= \int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4 \cdot 4\,dt$

$\Rightarrow ds = \|r'(t)\|\,dt$

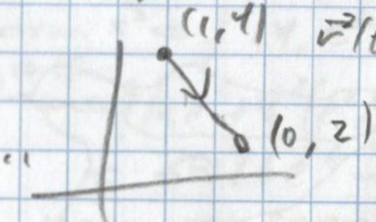
$= \sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt$

$= 4\,dt$

Line Integrals (w/ respect to x or y):

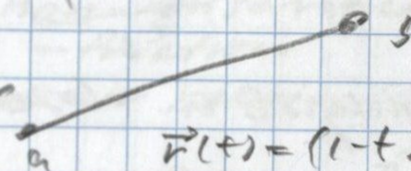
$\int_C \sin(\pi y)\,dy + yx^2\,dx$

$= \int_0^1 \sin(\pi(4-2t))(-2)\,dt + \int_0^1 (4-2t \ldots$



$\vec{r}(t) = (1-t)\begin{bmatrix}1\\4\end{bmatrix} + t[$

Line Integrals (w/ respect to vector fields):



$\vec{r}(t) = (1-t)\vec{a} + t\vec{b}$

$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

ex.

[Lecture Notes] 11.3.23

ex. Is this vector field $\vec{F}(x,y) = \begin{bmatrix} ye^x + \sin y \\ e^x + x\cos y + y^3 \end{bmatrix} \begin{matrix} P \\ Q \end{matrix}$ conservative on $\mathbb{R}^2$? If so, find (a) potential function $f(x,y)$. (ie. $\vec{F} = \nabla f$) ($P = f_x$, $Q = f_y$)

$f_{xy} = f_{yx} \Rightarrow P_y \overset{?}{=} Q_x \Rightarrow e^x + \cos y \overset{?}{=} e^x + \cos y$ ✓ ∴ conservative

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} ye^x + \sin y \\ e^x + x\cos y + y^3 \end{bmatrix} \rightsquigarrow f = \int ye^x + \sin y \, dx = \underbrace{ye^x + x\sin y + g(y)}$$

$\cancel{e^x} + \cancel{x\cos y} + g'(y) = \cancel{e^x} + \cancel{x\cos y} + y^3 \Rightarrow g'(y) = y^3 \Rightarrow g(y) = \frac{1}{4}y^4 \ (+C)$

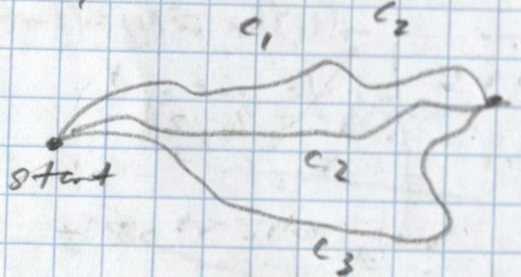
so $f = ye^x + x\sin y + \frac{1}{4}y^4$. (a potential function)

* conservative = gradient = exists $f(x,y)$ such that $\vec{F} = \nabla f$.

The Fundamental Theorem of Line Integrals:

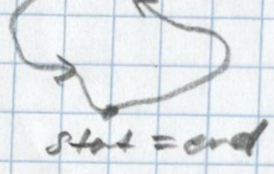
* $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

* line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \Rightarrow$ conservative



line integrals are usually dependent on path

* line integrals on closed paths = 0



$\oint_C = 0$

ex: Compute $\int_C (\ln y + 2xy^3)dx + (3x^2y^2 + \frac{x}{y})dy$ where

C has the parametric equations:

$$x = \frac{1}{2}t^2 + 2, \quad y = e^t(1 + 2t - t^2), \quad 0 \le t \le 2$$

$\int_C = \int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \begin{bmatrix} \ln y + 2xy^3 \\ 3x^2y^2 + \frac{x}{y} \end{bmatrix}\begin{matrix} P \\ Q \end{matrix}$.

Shortcut: hope $\vec{F}$ is conservative:

$P_y \stackrel{?}{=} Q_x \iff \frac{1}{y} + 6xy^2 \stackrel{?}{=} \frac{1}{y} + 6xy^2$ ✓ ∴ conservative

now find potential function f:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} \stackrel{\text{set}}{=} \begin{bmatrix} \ln(y) + 2xy^3 \\ 3x^2y^2 + \frac{x}{y} \end{bmatrix} \rightsquigarrow f = \int \ln y + 2xy^3 \, dx = x\ln y + x^2y^3 + g(y)$$

$\frac{x}{y} + 3x^2y^2 + g'(y) = 3x^2y^2 + \frac{x}{y} \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0 \; (+C)$

$\therefore f = x\ln(y) + x^2y^3$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \underset{\uparrow}{=} f(\underbrace{\vec{r}(b)}_{t=2}) - f(\underbrace{\vec{r}(a)}_{t=0})$$

fundamental theorem of line integrals

$= f(4, e^2) - f(2, 1)$

$= 4\ln e^2 + 4^2(e^2)^3 - (2\ln 1 + 2^2 1^3)$

$= 8 + 16e^6 - 4 = 4 + 16e^6$.

e.g. conservative vector fields:
- gravitational force
- electrical force

non-conservative:
- friction
- air resistance

ex. Determine whether the vector field

$$\vec{F} = \langle \underbrace{e^x \sin yz}_{P}, \underbrace{ze^x \cos yz}_{Q}, \underbrace{ye^x \cos yz + 3z^2}_{R} \rangle$$

is conservative. If so, find a potential function. Then find $\int_C \vec{F} \cdot d\vec{r}$ where C is a smooth curve where given by $\vec{r}(t)$ that starts at $\vec{r}(0) = \langle 0, \frac{\pi}{2}, 1 \rangle$ and ends at $\vec{r}(\pi) = \langle 1, 0, 2 \rangle$.

$$\begin{cases} P_y \overset{?}{=} Q_x \checkmark \ ze^x \cos(yz) \\ P_z \overset{?}{=} R_x \checkmark \ ye^x \cos(yz) \\ Q_z \overset{?}{=} R_y \checkmark \ e^x \cos(yz) - yze^x \sin(yz) \end{cases}$$

now find f where $\vec{F} = \nabla f$.

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \begin{matrix} \rightsquigarrow f = \int e^x \sin(yz)\,dx = e^x \sin(yz) + \overbrace{g(y,z)}^{\text{constant of integration}} \\ \rightsquigarrow ze^x \cos(yz) + \frac{\partial}{\partial y} g(y,z) = ze^x \cos(yz) \end{matrix}$$

$$\Rightarrow \frac{\partial}{\partial y} g(y,z) = 0 \Rightarrow g(y,z) = 0 + \underbrace{h(z)}_{\text{constant of integration}}$$

$\therefore f = e^x \sin(yz) + h(z)$

$\rightarrow ye^x \cos(yz) + h'(z) = ye^x \cos(yz) + 3z^2$

$\Rightarrow h'(z) = 3z^2 \rightarrow h(z) = z^3 (+ c)$

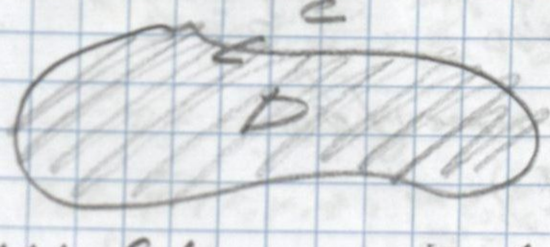
$\therefore f = e^x \sin(yz) + z^3$. Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(1,0,2) - f(0, \tfrac{\pi}{2}, 1)$$

$$= (e^1 \sin(0 \cdot 2) + 2^3) - (e^0 \sin(\tfrac{\pi}{2} \cdot 1) \cdot 1^3) = 8 - (1+1) = 6.$$

Lecture Notes | 11.6.23

Greens Theorem:



" $\oint$ on ∂D "

* Let C be a positively oriented piecewise smooth, simple closed curve in the plane which bounds a region D. If P and Q have continuous first partials on a region containing D. Then,

* $$\int_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Notations for a closed curve C,

* $\oint_C P\,dx + Q\,dy$ or $\oint_{\partial D} P\,dx + Q\,dy$ — ∂D: same "del" character, different meaning

to imply that C has positive orientation. Recall writing ∂D for boundary of D implies ∂D has positive orientation.

We can then rewrite Greens Theorem as:

* $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy.$$

ex. Compute $\oint_C \overbrace{(y+e^x)}^{P}dx + \overbrace{(2x+\cos y^2)}^{Q}dy$ where C is the boundary (oriented positively) of the region bounded by $y=x^2$ and $x=y^2$.

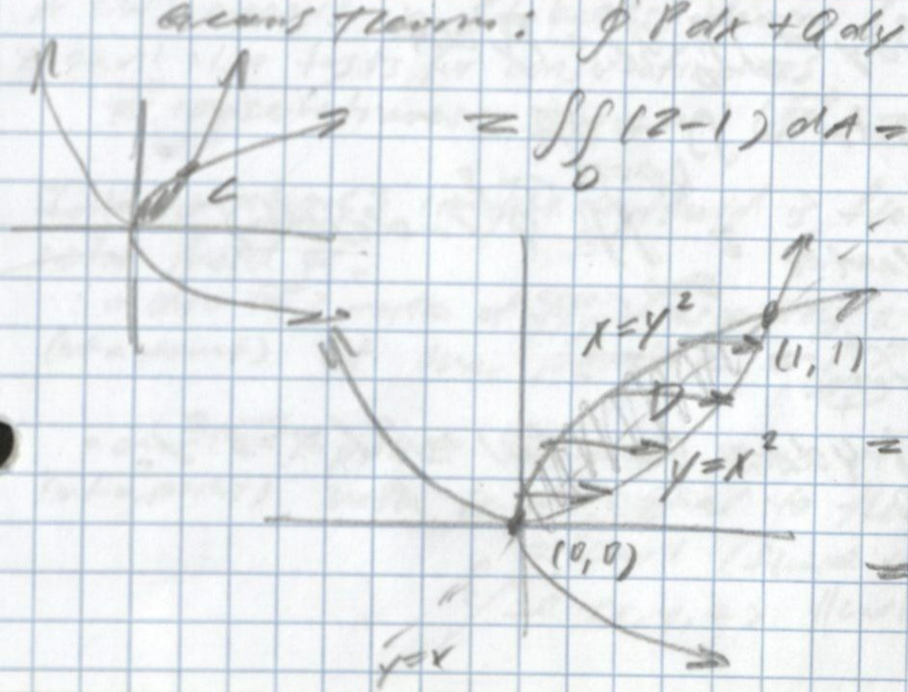
Greens Theorem: $\oint P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$

$$= \iint_D (2-1)\,dA = \iint_D 1\,dA$$

$$= \int_{y=0}^{y=1}\int_{x=y^2}^{x=\sqrt{y}} 1\,dx\,dy$$

$$= \int_{y=0}^{y=1} \sqrt{y} - y^2\,dy$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$



Area w/ Green's Theorem:

* $\text{Area}(D) = \iint_D 1 \, dA = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$

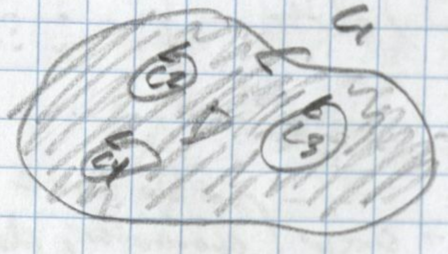
want to write $1 = Q_x - P_y$ for some $\begin{bmatrix} P \\ Q \end{bmatrix}$

(eg. $\begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} -y \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{y}{2} \\ \frac{x}{2} \end{bmatrix}$)

* If $D = D_1 \cup D_2$

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

Regions with Holes:

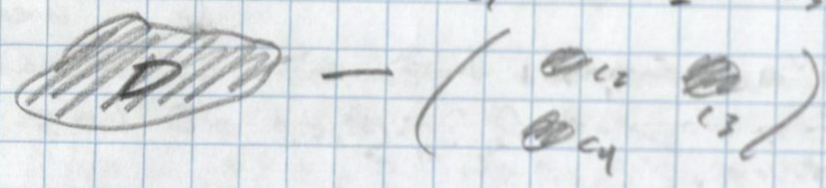


Green's Theorem!

$\iint_D \ldots =$

$\oint_{C_1} - \left(\oint_{C_2} + \oint_{C_3} + \oint_{C_4}\right)$

$D =$



* positive orientation $=$ counterclockwise (if D inside C)

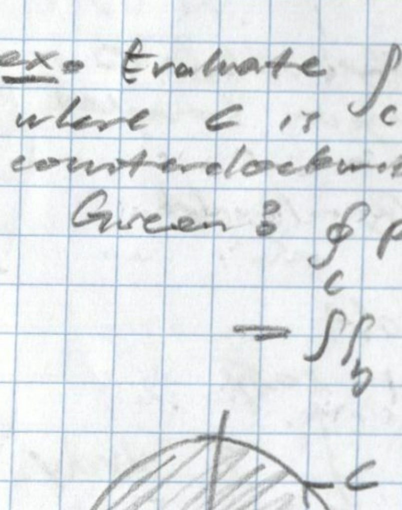
$\neq$ clockwise (if D outside C)

ex: Evaluate $\int_C \overbrace{(3y - e^{\sin x})}^{P}\,dx + \overbrace{(7x + \sqrt{y^4+1})}^{Q}\,dy$ where C is the circle $x^2+y^2=9$ oriented in counterclockwise fashion

Green: $\oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

$= \iint_D (7-3)\,dA = \iint 4\,dA$

polar coordinates $= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3} 4\,(r\,dr\,d\theta)$



$= \ldots = 36\pi.$

or $\iint_D 4\,dA = 4\,\text{area}(D) = 4(\pi \cdot 3^2).$

Curl and Divergence:
operator "del" ∇: or "grad" or "nabla"

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$* \ \nabla f = \text{grad} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Let $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ be a vector field:

Divergence of $\vec{F}$:

(real valued function)

$$* \ \text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Curl of $\vec{F}$:

(vector field on $\mathbb{R}^3$)

$$* \ \text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F} \Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

* notice to find the curl of $\vec{F}$, $\vec{F}$ must be a 3D vector field

ex. Find the curl and divergence of:

$$\vec{F} = \langle \underbrace{xy^2z^3}_{P}, \underbrace{x^3yz^2}_{Q}, \underbrace{x^2y^3z}_{R} \rangle$$

$$\text{div} \vec{F} = P_x + Q_y + R_z = y^2z^3 + x^3z^2 + x^2y^3$$

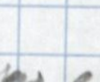
$$\text{curl} \vec{F} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} = \begin{bmatrix} 3x^2y^2z - 2x^3yz \\ 3xy^2z^2 - 2xy^3z \\ 3x^2yz^2 - 2xyz^3 \end{bmatrix}$$

* divergence appears in the divergence theorem (3D Green's Theorem)
* curl appears in Stokes's theorem (another more powerful 3D Green's theorem)
* curl also tests for conservativeness

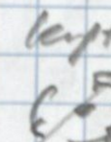
$\vec{F}$ conservative: $\Leftrightarrow$ $\text{curl}(\vec{F}) = \vec{0}$

Interpretations: imagine a fluid is flowing according to the vector field $\vec{F}$.

• $\text{div} \vec{F}$: rate of fluid escaping a small sphere centered (at a point) at the point

 (x,y,z) *infinitely small sphere

• $\text{curl} \vec{F}$: points along the axis of rotation of the fluid, (at a point) with length equal to the rate of rotation



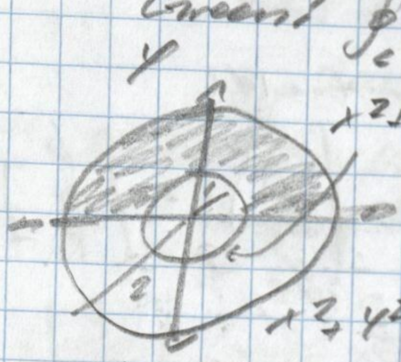
$\vec{F}$ curl (fluid rotating counterclockwise)

(x,y,z) $\|\text{curl} \vec{F}\|$ = length of curl $\vec{F}$.

[Lecture Notes] 11.8.23

ex. Evaluate $\int_C \overbrace{y^2}^{P}dx + \overbrace{3xy}^{Q}dy$ where C is the positively oriented boundary of the region in the upper-half-plane trapped between $x^2+y^2=1$, $x^2+y^2=4$

Green's $\oint_C P\,dx+Q\,dy = \iint_D Q_x - P_y\,dA = \iint_D 3y-2y\,dA = \iint_D y\,dA$



$0\le\theta\le\pi$

$1\le r\le 2$

polar $\Rightarrow \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} \cancel{r^2\sin\theta\,dr\,d\theta}$

$= \int_0^\pi \left[\frac{r^3}{3}\sin\theta\right]_1^2 d\theta = \int_0^\pi \frac{7}{3}\sin\theta\,d\theta = \frac{14}{3}.$

ex. Compute divergence and curl of $\vec{F} = \langle x,y,z\rangle$ and $\vec{F} = \langle x,-z,y\rangle$

$\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$, $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}\cdot\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = P_x + Q_y + R_z$

$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$

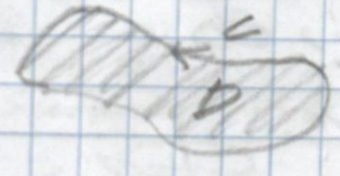
$\vec{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{matrix} P \\ Q \\ R \end{matrix} \Rightarrow \operatorname{div}\vec{F} = 1+1+1 = 3 \Rightarrow \operatorname{curl}\vec{F} = \begin{bmatrix} 0-0 \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

i.e. $\vec{F}$ is conservative/irrotational

$\vec{F} = \begin{bmatrix} x \\ -z \\ y \end{bmatrix}\begin{matrix} P \\ Q \\ R \end{matrix} \Rightarrow \operatorname{div}\vec{F} = 1-0+1 = 1 \Rightarrow \operatorname{curl}\vec{F} = \begin{bmatrix} 1-(-1) \\ 0-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

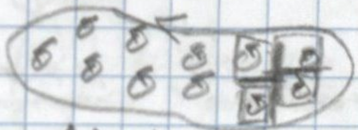
* $\operatorname{curl}(\nabla f) = \vec{0}$ (since ∇f is conservative)
* $\operatorname{div}(\operatorname{curl}\vec{F}) = \vec{0}$

Why is Green's Theorem true? Imagine a fluid in the plane is acted upon a force field $\vec{F}(x,y) = \begin{bmatrix} P \\ Q \end{bmatrix}$.



Green: $\underbrace{\oint_C P\,dx + Q\,dy}_{\text{work, i.e. net circulation along } C} = \underbrace{\iint_D Q_x - P_y \, dA}_{\text{integral of circulation on } D}$

$=$



integral of circulation on D

$Q_x - P_y$ = rate fluid is rotating counterclockwise since $\operatorname{curl} \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q_x - P_y \end{bmatrix}$.

<u>Parametric Surfaces:</u>

Suppose we have a surface S in $\mathbb{R}^3$. Since a surface is inherently 2-dimensional, it requires 2 variables to parametrize. A parametrization of S looks like:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D$$

where D is the domain

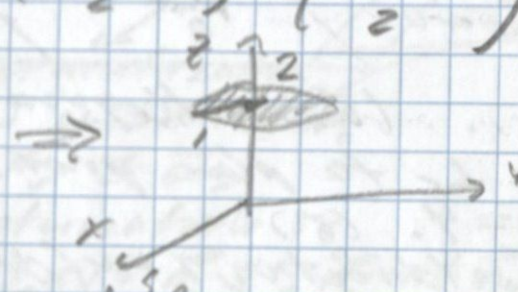
<u>ex.</u> Identify the sketch parametrized by:

a) $\vec{r}(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ 2 \end{bmatrix}$, $0 \le u \le 1$, $0 \le v \le 2\pi$

$\Rightarrow \vec{r} = \begin{bmatrix} u\cos v \\ u\sin v \\ 2 \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 2 \end{bmatrix}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

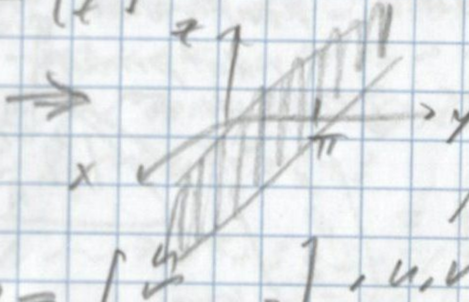
cylindrical coordinates

$\Rightarrow$



b) $\vec{r}(s,t) = \begin{bmatrix} s \\ \pi \\ t \end{bmatrix}$, $s, t \in \mathbb{R}$ $\Rightarrow$ just $y = \pi$

equation $\Rightarrow$

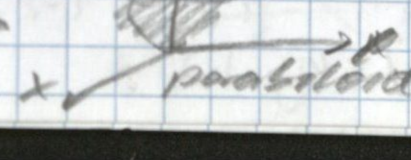


or in cylindrical coordinates $z = r^2$, so: $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$

c) $\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ u^2+v^2 \end{bmatrix}$, $u, v \in \mathbb{R}$ $\Rightarrow x = u$, $y = v$, $z = u^2 + v^2 = x^2 + y^2$

i.e. $z = x^2 + y^2$.

equations $\Rightarrow$

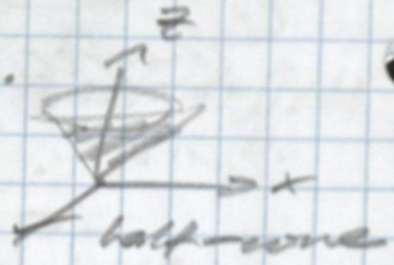


Since z is completely determined by x and y, we can let x and y be parameters from

$\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$, $(x,y) \in D$

ex. parametrize the surface $z = 3\sqrt{x^2+y^2}$.

$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ 3\sqrt{x^2+y^2} \end{bmatrix}, \quad (x,y) \in \mathbb{R}$$



or, we could use cylindrical coordinates

$$\vec{s}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 3r \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z = 3\sqrt{x^2+y^2} = 3\sqrt{r^2} = 3r \end{matrix} \quad (r \ge 0,\ 0 \le \theta \le 2\pi)$$

$\boxed{\text{Lecture Notes}}$ 11/10/23

ex. Parametrize the sphere $x^2+y^2+z^2 = 81$

spherical coordinates: $\rho = 9$

$x = r\cos\theta = (\rho\sin\phi)\cos\theta = 9\sin\phi\cos\theta$

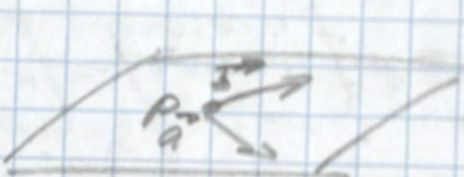
$y = r\sin\theta = (\rho\sin\phi)\sin\theta = 9\sin\phi\sin\theta$

$z = \rho\cos\phi = 9\cos\phi$

$$\vec{s}(\theta,\phi) = \begin{bmatrix} 9\sin\phi\cos\theta \\ 9\sin\phi\sin\theta \\ 9\cos\phi \end{bmatrix}, \quad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}$$

parametrizations of planes:

* $\vec{r}(s,t) = \vec{P} + s\vec{a} + t\vec{b}$
  where $\vec{a}$ and $\vec{b}$ are nonzero and non parallel



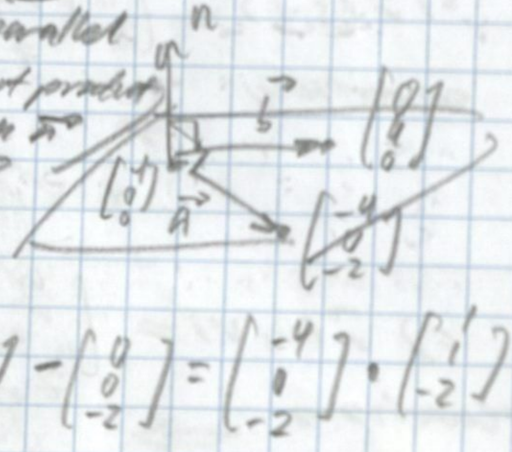
($\vec{a}, \vec{b}$ orthogonal to $\vec{n}$)

ex. find a parameterization for the plane given by the equation $x+y-2z=4$.

$\vec{r}(u,v) = \text{point} + u(\text{direction}_1) + v(\text{direction}_2)$ ($\vec{a}$)   ($\vec{b}$)

$$= \begin{bmatrix} 4-4u-4v \\ 4u \\ -2v \end{bmatrix}, \quad u,v \text{ in } \mathbb{R} \quad \text{nonparallel}$$

normal $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \vec{a} \times \vec{b}$ (dot product with $\vec{n}$ = 0)



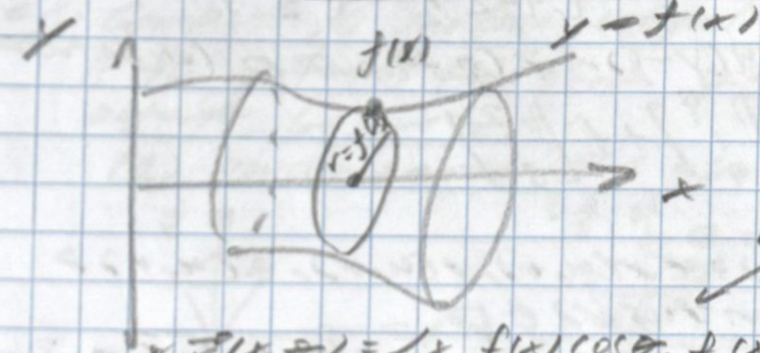
$\text{point}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ $\text{point}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

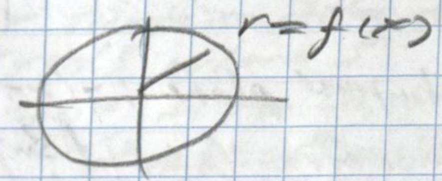
$\vec{a} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0$ $\vec{b} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0$

or solve for z in terms of x, y

$$z = \frac{x+y-4}{2} \rightsquigarrow \vec{s}(x,y) = \begin{bmatrix} x \\ y \\ \frac{x+y-4}{2} \end{bmatrix},\ x, y \text{ in } \mathbb{R}.$$

<u>Surfaces of Revolution:</u>





polar coordinates in yz-plane

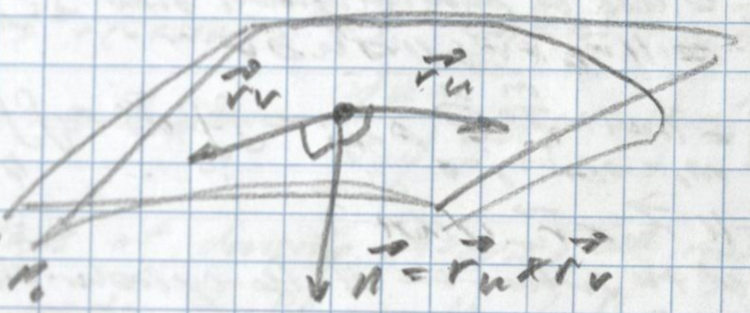
* $\vec{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$, $0 \le \theta \le 2\pi$

<u>Tangent Planes to Surfaces:</u>

* if $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
  and $P_0 = \vec{r}(u_0, v_0)$, then we can
  C_u: $\vec{r}(u, v_0)$ hold v constant at v_0
  C_v: $\vec{r}(u_0, v)$ hold u constant at u_0

the two tangent vectors at P_0 are:

* $\vec{r}_u = \frac{\partial \vec{r}}{\partial u}(u_0, v_0)$
* $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}(u_0, v_0)$



* we call a parametrization $\vec{r}(u,v)$ of S smooth at P_0 if $\vec{r}_u \times \vec{r}_v \ne \vec{0}$.

ex. A parameterization is
$$T_{P_0} S(s,t) = \vec{P_0} + s\vec{r}_u + t\vec{r}_v.$$

Find the tangent plane to the surface parametrized by $\vec{r}(u,v) = \langle u^2 - v^2, u+v, u^2+3v \rangle$ at the point $(3, 1, 1)$.

First, find (u, v) corresponding to $(3, 1, 1)$:

① $u^2 - v^2 = 3 \rightsquigarrow u^2 = v^2 + 3$ (or ①: $(u-v)\underbrace{(u+v)}_{=1} = 3$)
② $u + v = 1$
③ $u^2 + 3v = 1 \rightsquigarrow v^2 + 3 + 3v = 1 \rightsquigarrow v^2 + 3v + 2 = 0 \rightsquigarrow (v+2)(v+1) = 0$

$\rightsquigarrow \cancel{v = -2}$, $\boxed{v = -1}$

② $u = 1 - v \rightsquigarrow \cancel{u = 3}$, $\boxed{u = 2}$

fails ③

$\vec r = \begin{bmatrix} u^2 - v^2 \\ u+v \\ u^2+3v\end{bmatrix}$, $\vec r_u = \begin{bmatrix}2u\\1\\2u\end{bmatrix} \overset{(2,-1)}{=} \begin{bmatrix}4\\1\\4\end{bmatrix}$, $\vec r_v = \begin{bmatrix}-2v\\1\\3\end{bmatrix} \overset{(2,-1)}{=} \begin{bmatrix}2\\1\\3\end{bmatrix}$

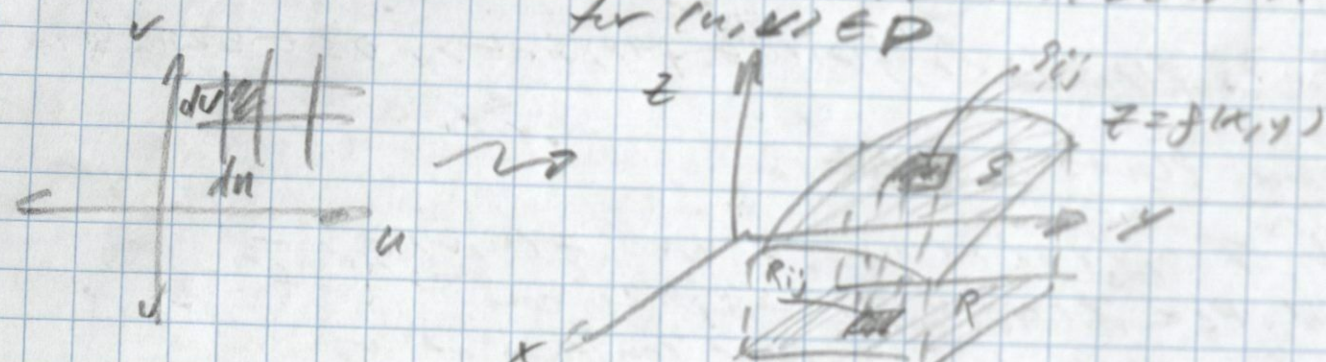
normal vector: $\vec r_u \times \vec r_v = \begin{bmatrix}4\\1\\4\end{bmatrix} \times \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}3-4\\8-12\\4-2\end{bmatrix} = \begin{bmatrix}-1\\-4\\2\end{bmatrix}$

$\Rightarrow$ tangent plane: $-(x-3)-4(y-1)+2(z-1)=0$ (equation)

$\Rightarrow$ parametric form: $\begin{bmatrix}3\\1\\1\end{bmatrix} + s\begin{bmatrix}4\\1\\4\end{bmatrix} + t\begin{bmatrix}2\\1\\3\end{bmatrix}$

<u>Surface Integrals:</u>

suppose we have $\vec r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for $(u,v)\in D$



$*$ area $= \|\vec r_u \times \vec r_v\|\, du\, dv = dS$

analogous to |Jacobian|

ΔS_{ij} = area of parallelogram over S_{ij}
$= \|\vec r_u \times \vec r_v\| \Delta u \Delta v$

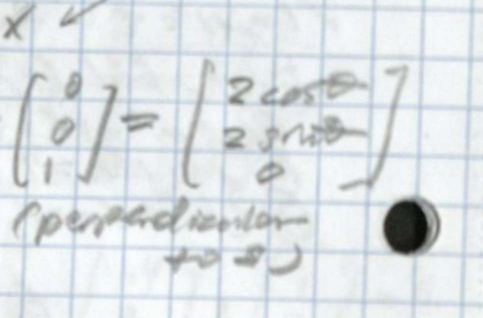
$*$ $A(S) = \lim\limits_{m,n\to\infty} \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} \Delta S_{ij} = \iint_S dS = \iint_D \|\vec r_u \times \vec r_v\|\, dA$

$*$ $dS = \|\vec r_u \times \vec r_v\|\, dA$

<u>ex.</u> Find the surface area of the cylinder described by $\overbrace{x^2+y^2}^{r^2} = 4$, $0\le z\le 3$

① parametrize: cylindrical coordinates: $r^2=4$, $(r=2)$

$\vec S(\theta,z) = \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}r\cos\theta\\r\sin\theta\\z\end{bmatrix} = \begin{bmatrix}2\cos\theta\\2\sin\theta\\z\end{bmatrix}$, $0\le\theta\le 2\pi$, $0\le z\le 3$



② integrate: $A(S) = \iint_S 1\, dS = \iint\limits_{\substack{0\le\theta\le2\pi\\0\le z\le3}} \|\vec S_\theta \times \vec S_z\|\, dA$

$\vec S_\theta = \begin{bmatrix}-2\sin\theta\\2\cos\theta\\0\end{bmatrix}$, $\vec S_z = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\rightarrow$ $\vec S_\theta \times \vec S_z = \begin{bmatrix}-2\sin\theta\\2\cos\theta\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\cos\theta\\2\sin\theta\\0\end{bmatrix}$ (perpendicular to z)

$\|\vec S_\theta \times \vec S_z\| = 2$

$\Rightarrow A(S) = \iint\limits_{\substack{0\le\theta\le2\pi\\0\le z\le3}} 2\, dA = 12\pi.$

Lecture Notes 11.13.23

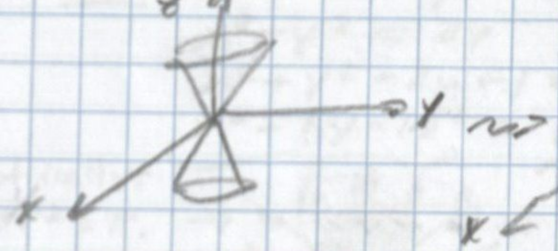
Surface Integrals:

$$* \iint_S f\,dS = \iint_D f(\vec{r}(u,v))\underbrace{\|\vec{r}_u \times \vec{r}_v\|\,dA}$$

scalar surface integral

ex: Compute the surface integral $\iint_S xyz\,dS$ where S is the piece of the cone $z^2 = \underbrace{x^2+y^2}_{r^2}$ in the first octant, below $z=1$. $\Rightarrow z^2 = r^2 \Rightarrow z = r$

(1) paramaterize S : use cylindrical coordinates



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r \end{bmatrix}, \quad 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}$$

$(\vec{q}(r,\theta))$

(2) integrate:

$$\iint_S xyz\,dS = \iint_{\substack{0\le r\le 1 \\ 0\le\theta\le\frac{\pi}{2}}} (r\cos\theta)(r\sin\theta)\,r\,(\|\vec{q}_r \times \vec{q}_\theta\|)\,dA$$

$$\vec{q}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix},\ \vec{q}_\theta = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix},\ \vec{q}_r \times \vec{q}_\theta = \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{bmatrix}$$

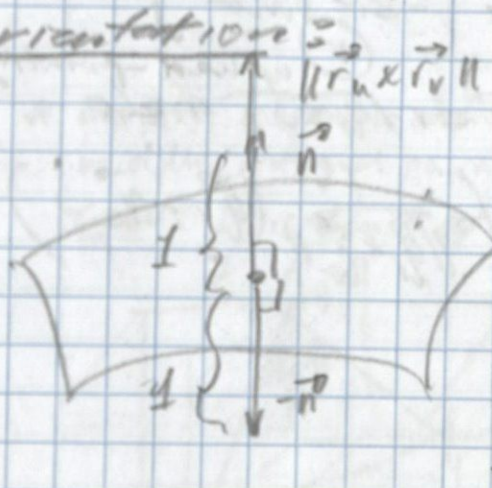
$$\|\vec{q}_r \times \vec{q}_\theta\| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = \sqrt{2r^2} = \sqrt{2}\,r$$

$$\Rightarrow \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1} \sqrt{2}\,r^4 \cos\theta\sin\theta\,dr\,d\theta = \frac{\sqrt{2}}{5}\int_0^{\pi/2} \cos\theta\sin\theta\,d\theta = \frac{\sqrt{2}}{5}\left(\frac{1}{2}\right)$$

Application: Center of Mass of density $\rho(x,y,z)$ and mass m

$$* (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{m}\iint_S x\rho\,dS,\ \frac{1}{m}\iint_S y\rho\,dS,\ \frac{1}{m}\iint_S z\rho\,dS\right)$$

Orientation:



* if a surface is orientable, it has exactly two orientations

* an orientation on a surface S is a choice of a continuous unit normal vector field on S.

* a cylinder is orientable, while a moebius strip is not

If the 2 orientations are $\vec{n}_1$ and $\vec{n}_2$, then:

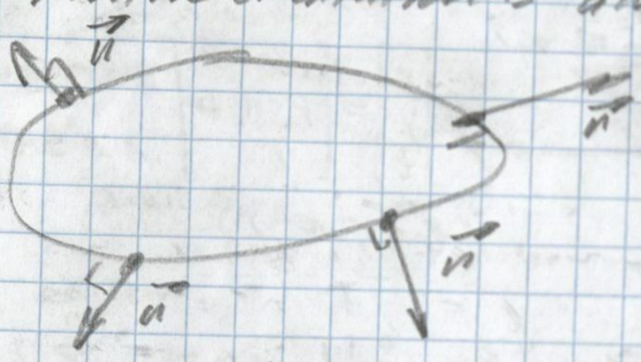
* $\vec{n}_1 = -\vec{n}_2$

If S is parameterized by $\vec{r}(u,v)$, then:

* $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ and $-\vec{n} = \dfrac{\vec{r}_v \times \vec{r}_u}{\|\vec{r}_v \times \vec{r}_u\|}$

If a surface is closed, then:

* the positive orientation is always outward

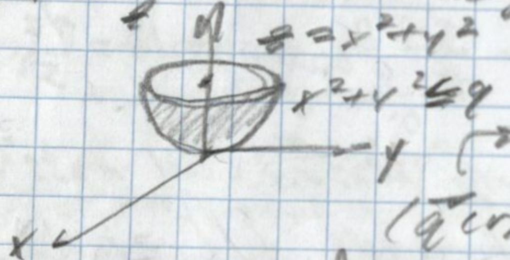


* If the surface is not closed, there is no canonical orientation

(z-component > 0) (normal vector)

Ex - Find the upward pointing orientation on the surface which is the graph of $f(x,y) = x^2+y^2 = z$ over $x^2+y^2 \le 9$.

parameterize S using cylindrical coordinates



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{bmatrix}, \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

$(\vec{q}(r,\theta))$

$$\vec{q}_r \times \vec{q}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 2r \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{bmatrix}$$

upward-pointing

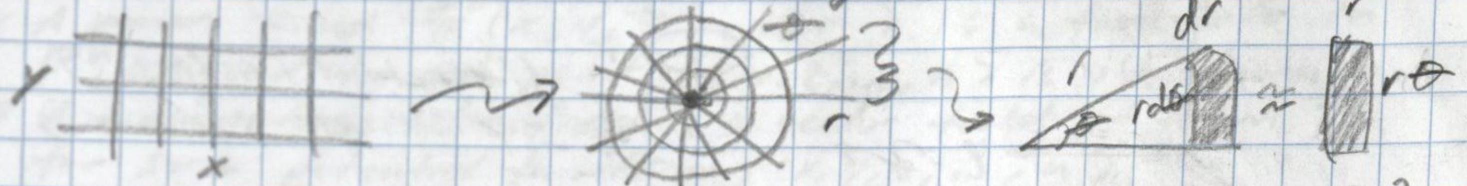
$$\|\vec{q}_r \times \vec{q}_\theta\| = \sqrt{(-2r^2\cos\theta)^2 + (-2r^2\sin\theta)^2 + r^2} = r\sqrt{4r^2+1} \quad r \ge 0$$

$$\therefore \vec{n} = \frac{1}{r\sqrt{4r^2+1}} \begin{bmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{bmatrix}. \quad \left(= \frac{1}{\|\vec{q}_r \times \vec{q}_\theta\|}(\vec{q}_r \times \vec{q}_\theta)\right)$$

$\therefore$ for $-\vec{n}$, multiply answer by -1.

[Independent Notes] Exam 3 11.14.23

Standard 12: Calculate double integrals using polar coordinates



cartesian grid → polar coordinates

$* \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$* \; dA = dx\,dy \implies dA = r\,dr\,d\theta$

$* \; r^2 = x^2 + y^2 \quad * \tan\theta = y/x$

$0 \le r \le R^2$

$0 \le \theta \le 2\pi$

Case: area of region enclosed by:

$x^2 + y^2 = 2y$

$x^2 + y^2 - 2y + 1 = 0 + 1$ ← complete the square

$x^2 + (y-1)^2 = 1$

Geometrically! $A = \pi r^2 = \pi$.

$D = \pi \quad r = 2\sin\theta$

$r^2 = 1 \implies r = 1$, but since not centered at $(0,0)$, bounds:

$x^2 + y^2 = 2y \implies r^2 = 2r\sin\theta$

$\implies r = 2\sin\theta \implies 0 \le r \le \underline{2\sin\theta}$

and $0 \le \theta \le \underline{\pi}$. π

$$A = \int_{\theta=0}^{\pi}\int_{r=0}^{2\sin\theta} 1\, r\,dr\,d\theta = \int_0^{\pi} \frac{1}{2} r^2 \Big|_0^{2\sin\theta} d\theta = \int_0^{\pi} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\pi} 2\left(\frac{1-\cos(2\theta)}{2}\right) d\theta = \Big[\theta - 2\sin 2\theta\Big]_0^{\pi} = \pi - 0 = \pi.$$

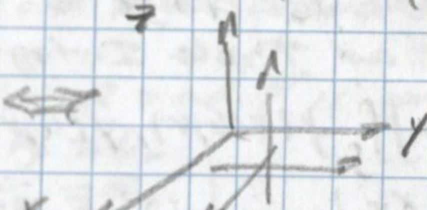
Standard 13: Calculate triple integrals over rectangular prisms and simple regions.

$$V(D) = \iiint_D 1\, dV \;;\; dV = dx\,dy\,dz \;;\; D = \underset{x}{[0,1]} \times \underset{y}{[1,2]} \times \underset{z}{[2,3]}$$

x: "back to front"

* y: "left to right" ⟺

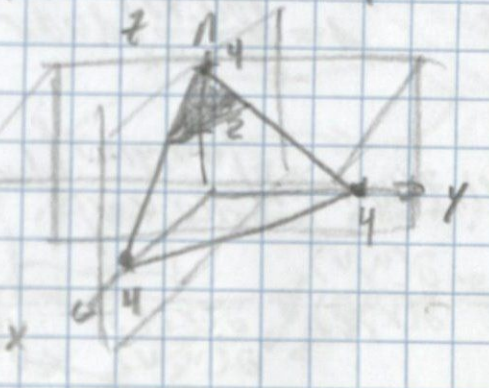
z: "bottom to top"



(any integration is in $\mathbb{R}^4$)

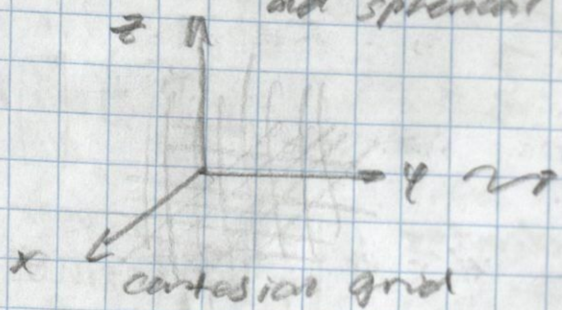
* outside integral cannot depend on any other variables and middle integral cannot depend on inside variable

Case: D is bound by $x=0$, $y=0$, $z=2$, and $x+y+z=4$



$$V(D) = \iiint_D 1\, dV = \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\int_{z=2}^{z=4-x-y} 1\, dz\,dy\,dx.$$

Standard 14: Calculate Triple Integrals using cylindrical and spherical coordinates

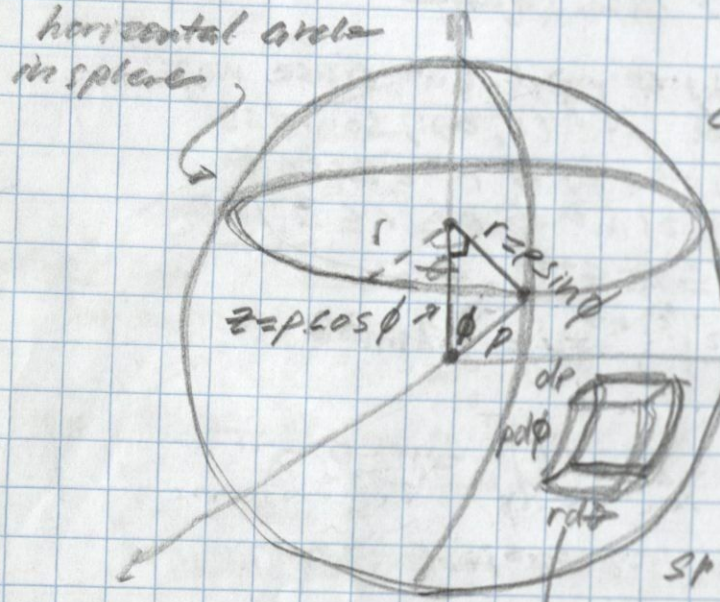


('stack' of polar coordinate systems)

cylindrical coordinate system

$dV = dx\,dy\,dz \Rightarrow dV = r\,dr\,d\theta\,dz$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix};\quad \begin{matrix} 0 \le r \le \mathbb{R}^2 \\ \theta \le 0 \le 2\pi \\ z \in \mathbb{R}^2 \end{matrix}$$



"rho"

↳ ρ = distance from origin $= \sqrt{x^2+y^2+z^2}$

"phi"

ϕ = angle from positive z-axis

θ = angle from positive x-axis

$$* \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\rho\sin\phi)\cos\theta \\ (\rho\sin\phi)\sin\theta \\ \rho\cos\phi \end{bmatrix};\quad \begin{matrix} 0 \le \rho \le \mathbb{R}^2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{matrix}$$

spherical coordinate system

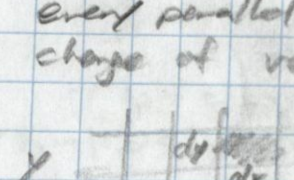
$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

Standard 15: Be able to use Change of Variables in both Double and Triple Integrals

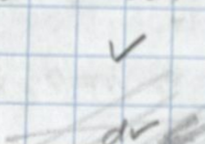
$* \iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$* \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$ ← Jacobian Matrix

every parallelogram becomes a rectangle under the right change of variables



$A = dx\,dy$



$A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$* \iiint_R f(x,y,z)\,dV = \iiint f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

$$* \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{array}{c} \begin{matrix} (+) & (-) & (+) \end{matrix} \\ \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \end{array}$$

$* \text{Inverse Jacobian} = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Standard 16: Calculate Line Integrals in both Scalar (ds) and Vector ($d\vec{s}$)

* A vector field $\vec{F}(x, y, z, \dots, n)$ is a function in $\mathbb{R}^n$ assigned to each point $(x, y, z, \dots, n)$ in its domain.
* A conservative vector field is a vector field of form ∇f for some potential function $f(x, y, z, \dots, n)$.

* $L = \int_S 1\, ds$, so * $\int_C f(x, y, \dots n)\, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}\,'(t)\|\, dt$

ds: arc length (unit-speed parametrization)

$\vec{r}(t)$: parametrization of C

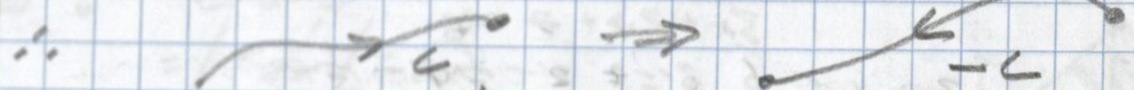
Case: C is parametrized by $r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, $a \le t \le b$

* $\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$

$$= \int_a^b P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t).$$

w/ respect to orientation

$\therefore$



* $\therefore W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \left(\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \right) \|\vec{r}\,'(t)\|\, dt$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_C \vec{F} \cdot d\vec{r}.$$

Standard 17: Use the Fundamental Theorem of Line Integrals

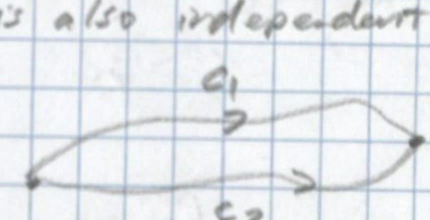
* if $\vec{F} = \langle P, Q \rangle$ or $\vec{F} = \langle P, Q, R \rangle$ is conservative, then $P = f_x$ and $Q = f_y$ or $P = f_x$, $Q = f_y$, and $R = f_z$ for some $f = \langle f_x, f_y \rangle$ or $f = \langle f_x, f_y, f_z \rangle$.

  if $\vec{F} = \langle x, y \rangle = \langle P, Q \rangle$ and $\vec{F}$ is conservative, then
  * $P_y = Q_x$.

  if $\vec{F} = \langle x, y, z \rangle = \langle P, Q, R \rangle$ and $\vec{F}$ is conservative, then
  * $\text{curl}\, \vec{F} = 0 \Longleftrightarrow \begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$.

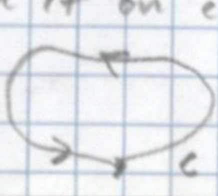
* and if conservative, $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$.

and is also independent of path, meaning:



$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

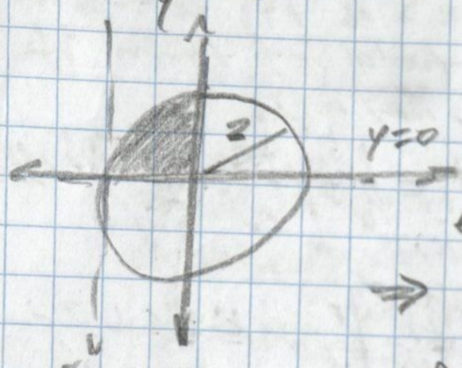
* and if on closed path:



$$\oint_C \nabla f \cdot d\vec{r} = 0$$

|Lecture Notes| Midterm 3 Review 11.15.23

S12. Rewrite $\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$ using polar coordinates



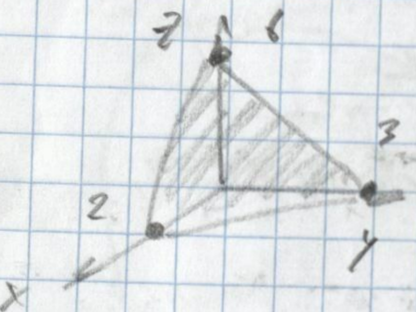
$-2 \le x \le 0$

$0 \le y \le \sqrt{4-x^2}$

$y^2 \le 4 - x^2 \Rightarrow x^2+y^2 \le 4$

$$\Rightarrow \int_{\theta=\frac{\pi}{2}}^{\theta=\pi}\int_{r=0}^{r=2} e^{-r^2}(r\,dr\,d\theta).$$

S13. Set up $\iiint_E f(x,y,z)\,dV$ as the iterated integral in order $dV = dx\,dy\,dz$, where E is the region in the first octant bounded by $3x+2y+z=6$



$3x+2y+z=6$

$$\Rightarrow \int_0^6\int_0^{3-\frac{1}{2}z}\int_0^{2-\frac{2}{3}y-\frac{1}{3}z} f(x,y,z)\,dx\,dy\,dz.$$

S14. Set up $\iiint_E x\,dV$ as iterated integral in spherical coordinates where E is in first octant and bounded by $x^2+y^2+z^2=4$ and above $z^2=x^2+y^2$.

$\rho^2 = 4 \Rightarrow \rho = 2$ $\quad x^2+y^2-z^2=0$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{2} \rho\sin\phi\cos\theta(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta).$$

or: $z^2 = x^2+y^2 \Rightarrow z^2 = r^2 \Rightarrow z = r$

$\Rightarrow \rho\cos\phi = \rho\sin\phi \Rightarrow \tan\phi = 1$

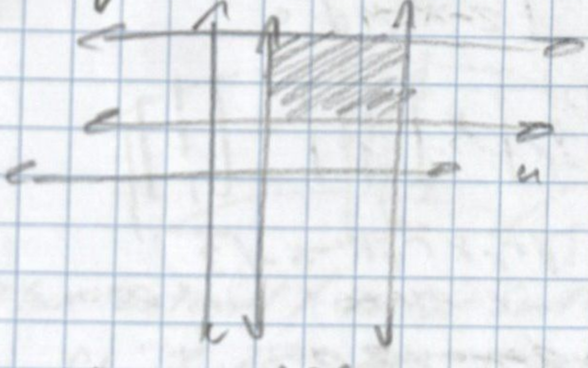
$\Rightarrow \phi = \frac{\pi}{4}.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \rho\cos\phi \end{bmatrix},\ r = \rho\sin\phi$$

515. Evaluate $\iint_D xy\,dA$ where D is the region in the first quadrant bounded by the curves

$\frac{y}{x}=1, \quad \frac{y}{x}=3, \quad xy=1, \quad xy=3$

$u=\frac{y}{x}, \quad v=xy$



$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \rightarrow \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = -\frac{y}{x} - \frac{y}{x} = -\frac{2y}{x} = -2u$$

$$\iint xy \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=\left|-\frac{1}{2u}\right| = \frac{1}{2u}} du\,dv = \int_1^3\int_1^3 v\left(\frac{1}{2u}\right) du\,dv = \ldots = 2\ln 3.$$

516. Evaluate the line integral $\int_C y\,ds$ where C is the portion of the circle $x^2+y^2=4$ in the first quadrant.

$x^2+y^2=4$ $\quad 0\le r\le 2$ $\quad \vec r(t)=\begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}, \quad 0\le t\le \frac{\pi}{2}$

$r^2=4$ $\quad 0\le \theta\le \frac{\pi}{2}$

$r=2$

$$\int_C y\,ds = \int_a^b f(\vec r(t))\|\vec r\,'(t)\|\,dt$$

$$= \int_0^{\frac{\pi}{2}} 2\sin t\,(2)\,dt = \int_0^{\frac{\pi}{2}} 4\sin t\,dt = -4\cos t\Big|_0^{\frac{\pi}{2}} = 4.$$

517. Evaluate the line integral $\int_C \vec F\cdot d\vec r$, where $\vec F = \begin{bmatrix} x^2y^3 \\ x^3y^2+\cos y \end{bmatrix}$ and C is parameterized by

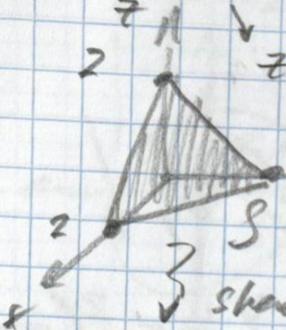
$\vec r(t) = \begin{bmatrix} t^{100}-2t \\ t^{100} \end{bmatrix}$ from $t=0$ to $t=1$.

if $\vec F=\nabla f$, then $\int_C \vec F\cdot d\vec r = f(\text{end of } C) - f(\text{start of } C)$

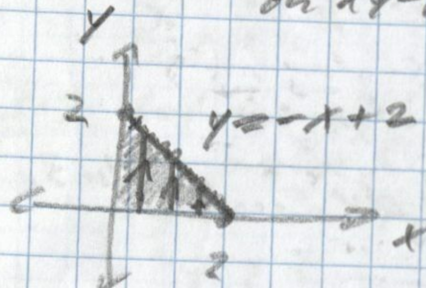
$\boxed{\text{Lecture Notes}}$ 11.17.23

ex. find $\iint_S xz\,dS$ where S is part of the plane $x+y+z=2$ in the $\underbrace{\text{first octant}}_{x,y,z\geq 0}$.

$z = 2-x-y$



shadow on xy-plane



$$\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2-x-y \end{bmatrix}$$

$$\vec{r}_x \times \vec{r}_y = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$x \geq 0$ $\quad \|\vec{r}_x \times \vec{r}_y\| = \sqrt{3}$

or $y \geq 0$

$z \geq 0 \Rightarrow 2-x-y \geq 0$

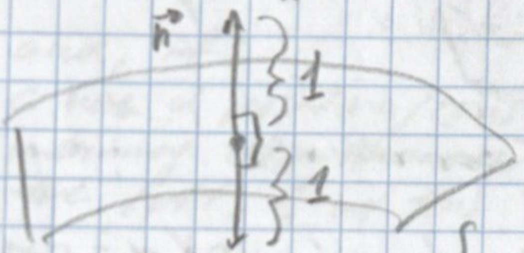
$$\Rightarrow \iint_S xz\,dS = \iint x(2-x-y)\,\|\vec{r}_x \times \vec{r}_y\|\,dx\,dy$$

$$= \int_0^2 \int_0^{2-x} x(2-x-y)(\sqrt{3})\,dx\,dy = \dots = \frac{2\sqrt{3}}{3}.$$

Flux Integrals:

* flux is measuring the rate at which something flows through a surface (derivative of vector field)

$\star\ d\vec{S} = \vec{n}\,dS = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}\|\vec{r}_u \times \vec{r}_v\|\,dA = (\vec{r}_u \times \vec{r}_v)\,dA$



$\vec{n} = \pm \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

$\vec{n}$: unit normal vector

* Flux Integral / Vector Surface Integral:

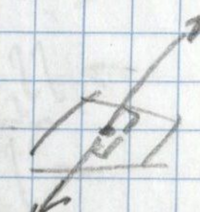
$$\star \iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{n})\,dS = \iint \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

scalar integral of $\vec{F}\cdot\vec{n}$

plug in

where S is parametrized by $\vec{r}(u,v)$ and $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

ex - Find the flux of $\vec{F} = \langle x, y, z\rangle$ across the helicoid S parametrized by $\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$ with upward orientation



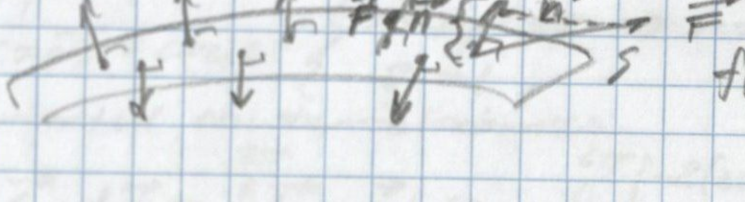
normal vector $\vec{r}_u \times \vec{r}_v = \begin{bmatrix}\cos v\\ \sin v\\ 0\end{bmatrix} \times \begin{bmatrix}-u\sin v\\ u\cos v\\ 1\end{bmatrix} = \begin{bmatrix}\sin v\\ -\cos v\\ u\end{bmatrix}$

upwards orientation: z-entry ≥ 0

$u \ge 0 \Rightarrow$ upward

$$\text{flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_{\substack{0\le u\le 1\\ 0\le v\le \frac{\pi}{2}}} \underbrace{\begin{bmatrix}u\cos v\\ u\sin v\\ v\end{bmatrix}}_{\vec{F}(\vec{r}(u,v))}\cdot\underbrace{\begin{bmatrix}\sin v\\ -\cos v\\ u\end{bmatrix}}_{\vec{r}_u\times\vec{r}_v} du\,dv = \int_0^{\pi/2}\int_0^1 vu\,du\,dv = \dots = \frac{\pi^2}{16}$$

* Interpretation: imagine $\vec{F}$ is the velocity field of a fluid, S is a mesh net which does not impede flow. Then the flux $\iint_S \vec{F}\cdot d\vec{S}$ is the net rate of flow across S.



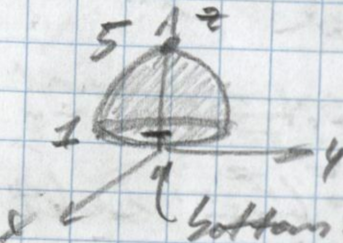
$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$

Ex: Compute $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \text{curl}(\vec{G})$, $\vec{G} = \langle \overset{P}{-2yz}, \overset{Q}{y}, \overset{R}{3xz} \rangle$, and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ w/ upwards orientation

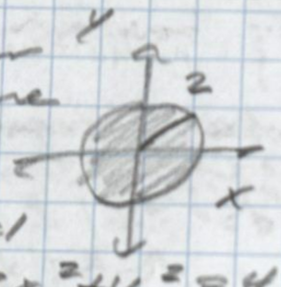
$$\text{curl}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \nabla \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 0 \\ -2y - 3 \\ 0 - (-2z) \end{bmatrix} = \begin{bmatrix} 0 \\ -2y - 3 \\ 2z \end{bmatrix}$$

Parametrize S: use cylindrical coordinates



Shadow in xy-plane



$$\vec{q}(r,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 5 - r^2 \end{bmatrix},$$

$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi.$$

bottom circle: $z = 1$

$$\Rightarrow 5 - x^2 - y^2 = 1 \Rightarrow x^2 + y^2 = 4$$

normal vector $\vec{q}_r \times \vec{q}_\theta$

$$= \begin{bmatrix} \cos\theta \\ \sin\theta \\ -2r \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ r \end{bmatrix}$$

$r \ge 0 \Rightarrow$ upward

$$\Rightarrow \iint \vec{F}\cdot d\vec{S} = \iint_D \begin{bmatrix} 0 \\ -2(r\sin\theta) - 3 \\ 2(5 - r^2) \end{bmatrix} \cdot \begin{bmatrix} 2r^2\cos\theta \\ 2r^2\sin\theta \\ r \end{bmatrix} dr\, d\theta$$

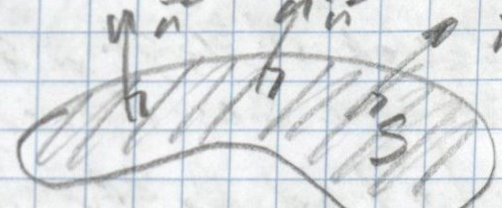
$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{r=2} (-2r\sin\theta - 3)(2r^2\sin\theta) + 2(5 - r^2)r \, dr\, d\theta$$

$$= \dots = 8\pi.$$

Lecture Notes 11.20.23

Stokes' Theorem:

Let S be a surface with boundary C and orientation $\vec{n}$.



and, if:
C has a positive/consistent orientation with S, meaning when traversing S, C should be on the left. C w/ this orientation is denoted ∂S

"Let S be an oriented piecewise-smooth surface which is bounded by a simple, closed, piecewise-smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ which is C^1 in an open region containing S." Then:

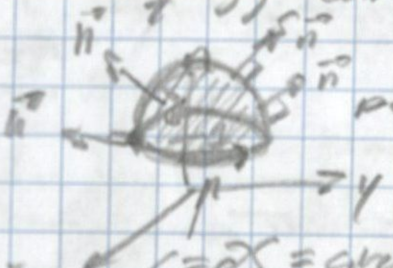
$$\int_{C = \partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot d\vec{S}$$

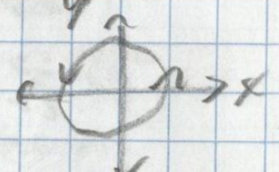
Line Integral — flux integral

Ex. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \text{curl } \vec{G}$.
$\vec{G} = \langle -2yz, y, 3x \rangle$ and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$ w/ upwards orientation

Stokes' Theorem:

$$\iint \text{curl}(\vec{G}) \cdot d\vec{S} = \int_{C = \partial S} \vec{G} \cdot d\vec{r}$$





parametrize C: $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \\ 1 \end{bmatrix}$, $0 \le \theta \le$ [illegible]

orient counter-clo[ckwise]

$C = \partial S = \text{circle} = \begin{cases} x^2 + y^2 = 4 \Rightarrow r = 2 \\ z = 1 \end{cases}$

oriented counter-clockwise

$$\int_C \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \vec{G} \cdot \vec{r}\,' \, d\theta = \int_0^{2\pi} \begin{bmatrix} -2(2\sin\theta)(1) \\ 2\sin\theta \\ 3(2\cos\theta) \end{bmatrix} \cdot \begin{bmatrix} -2\sin\theta \\ 2\cos\theta \\ 0 \end{bmatrix} d\theta$$

$$= \int_0^{2\pi} 8\sin^2\theta + 4\sin\theta\cos\theta \, d\theta = \dots = 8\pi.$$

Recall Green's Theorem: Surface S in xy-plane, then

* $\int_{\partial S} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r} = \iint_S Q_x - P_y \, dA$

* special case of Stokes' theorem: Same S, $\vec{F} = \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix}$

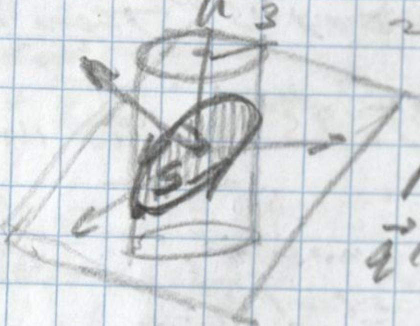
idea: parametrize S as $\vec{r}(x,y) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$;

$\vec{r}_x \times \vec{r}_y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $Q_x - P_y = \text{curl}\,\vec{F} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, 2z, 3y \rangle$ and C is the curve of intersection between $x + z = 5$ and $x^2 + y^2 = 9$, oriented counter-clockwise when viewed from above.

(1) Stokes':

$\vec{F} = \begin{array}{c} \begin{bmatrix} xy \\ 2z \\ 3y \end{bmatrix} \end{array}\begin{array}{c} P \\ Q \\ R \end{array}$ Stokes': $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$



$\rightarrow \text{curl}\,\vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} xy \\ 2z \\ 3y \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0-0 \\ 0-x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -x \end{bmatrix}$

parametrize S (cylindrical coordinates):

$\vec{q}(r,\theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 5 - r\cos\theta \end{bmatrix}$,

C = ellipse

S = inside of C oriented upward

$z = 5 - x = 5 - r\cos\theta$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$\vec{q}_r \times \vec{q}_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \\ -\cos\theta \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} r\sin^2\theta + r\cos^2\theta \\ r\sin\theta\cos\theta - r\sin\theta\cos\theta \\ r\cos^2\theta + r\sin^2\theta \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix}$

$r \ge 0$ $\therefore$ upward

$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_{(r,\theta)\text{-domain}} \begin{bmatrix} 1 \\ 0 \\ -r\cos\theta \end{bmatrix} \cdot \begin{bmatrix} r \\ 0 \\ r \end{bmatrix} dr\,d\theta$

$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} (r - r^2\cos\theta)\,dr\,d\theta = \ldots = 9\pi.$

(2) parametrize C: $\vec{r} = \begin{bmatrix} 3\cos\theta \\ 3\sin\theta \\ 5 - 3\sin\theta \end{bmatrix}$, $0 \le \theta \le 2\pi$

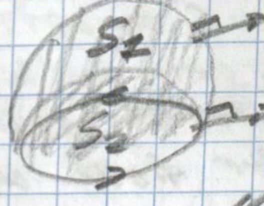
$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F} \cdot \vec{r}\,' \, d\theta$

$= \ldots = \int_0^{2\pi} -27\sin^2\theta\cos\theta + 6(5 - 3\cos\theta)\cos\theta + 27\sin^2\theta \, d\theta$

$= 9\pi.$

<u>A trick:</u> Suppose S_1 and S_2 have the same boundary C, and they have the same orientation on C, then:

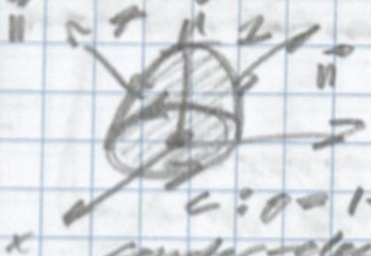
$$\star \iint_{S_1} \text{curl}\,\vec F\cdot d\vec S = \int_C \vec F\cdot d\vec r = \iint_{S_2} \text{curl}\,\vec F\cdot d\vec S$$



Lecture Notes

11.27.23

ex. Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec F\cdot d\vec S$ where $\vec F = \langle x^2\sin z, y^2, xy\rangle$ and S is part of the paraboloid $z = 1-x^2-y^2$ above the xy-plane, having upwards orientation



C: $0 = 1-x^2-y^2$, $z=0$
counter-clockwise
(the surface on left)
(upwards orientation)

Stokes: $\iint_S \text{curl}\,\vec F\cdot d\vec S = \int_C \vec F\cdot d\vec r$

$$= \int_0^{2\pi} \vec F(\vec r(\theta))\cdot \vec r\,'(\theta)\,d\theta$$

$$= \int_0^{2\pi} \begin{bmatrix}\cos^2\theta\sin 0\\ \sin^2\theta\\ \cos\theta\sin\theta\end{bmatrix}\cdot\begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix}d\theta$$

parametrize C: $\vec r(\theta) = \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix}$, $0\le\theta\le 2\pi$

$$= \int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta$$

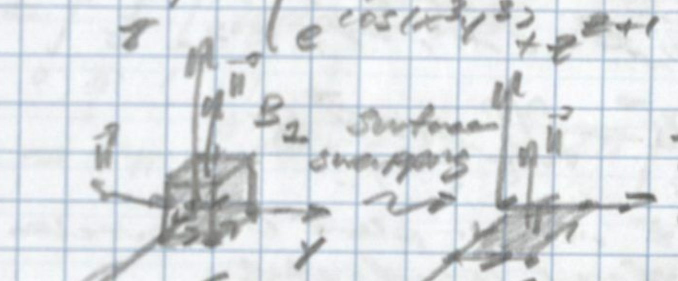
$u = \sin\theta$, $du = \cos\theta$

$$= \left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} = 0.$$

<u>Surface Swapping:</u>

<u>ex.</u> Let B_1 be the 5-sided boundary of the cube $0\le x,y,z\le 1$. Find $\iint_{B_1}\text{curl}\,\vec F\cdot d\vec S$ where

$\vec F = \begin{bmatrix}x^2y\\ xyz\\ e^{\cos(x^3y^3)}+z^{z+1}\end{bmatrix}$, excluding the bottom and oriented upward.



B_2
unit square
$0\le x,y\le 1$
$z=0$

$\therefore \iint_{B_1}\text{curl}\,\vec F\cdot d\vec S = \iint_{B_2}\text{curl}\,\vec F\cdot d\vec S$

$$\therefore \iint_{B_2}\text{curl}\,\vec F\cdot d\vec S = \iint_{\substack{0\le x\le 1\\ 0\le y\le 1}} \begin{bmatrix}R_y-Q_z\\ P_z-R_x\\ Q_x-P_y\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}dx\,dy$$

parametrize B_2: $\vec r(x,y) = \begin{bmatrix}x\\y\\0\end{bmatrix}$, $0\le x\le 1$, $0\le y\le 1$

$$= \iint Q_x - P_y\,dx\,dy$$

Normal vector $\vec r_x\times\vec r_y$: $\begin{bmatrix}1\\0\\0\end{bmatrix}\times\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ pointing up ✓

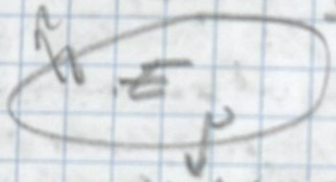
$0\le x\le 1$, $0\le y\le 1$, $yz - x^2 = -x^2$ when $z=0$

$$= \int_0^1\int_0^1 -x^2\,dx\,dy = \dots = -\frac{1}{3}.$$

The Divergence Theorem:

Let E be a bounded solid region in $\mathbb{R}^3$ with boundary S, where S consists of finitely many piecewise smooth, closed, orientable surfaces, each of which oriented with normals pointing away from E. Let $\vec{F}$ be a vector field which is C^1 on an open region containing E. Then,

$$* \underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{} = \underbrace{\iiint_E \operatorname{div}\vec{F}\, dV}_{}$$

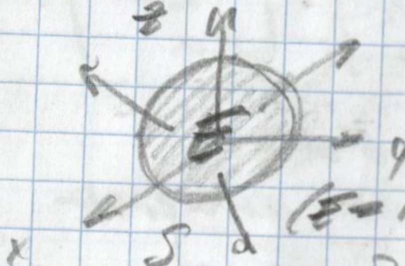


positively oriented: S oriented outward

if $\vec{F}$ is velocity field of a fluid, then rate of fluid crossing S

div = rate of fluid escaping a small ball centered at that point

ex: Compute the flux of $\vec{F} = \langle z, y, x\rangle$ across the sphere $x^2+y^2+z^2 = 89^2 = 7921$, where the sphere is oriented outward



(E = inside of S)

divergence theorem: flux of $\vec{F} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

$= \iiint_E 1\, dV = \text{volume}(E)$

$= \frac{4}{3}\pi \cdot 89^3$

or use spherical coordinates w/ $0\le\rho\le 89$, $0\le\theta\le 2\pi$, $0\le\phi\le\pi$.

$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}z + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}x$

$= 0 + 1 + 0 = 1$

ex: Compute $\iint_S \vec{F}\cdot d\vec{S}$ where S is the boundary of the solid bounded by $z = 4 - x^2 - y^2$ and the xy-plane, oriented positively, where $\vec{F} = \langle x^2, xy, z\rangle$.

divergence theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

note: S contains bottom as is whole boundary of shape

shadow of E in xy-plane is a circle ∴ use cylindrical coordinates

$$\therefore \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta$$

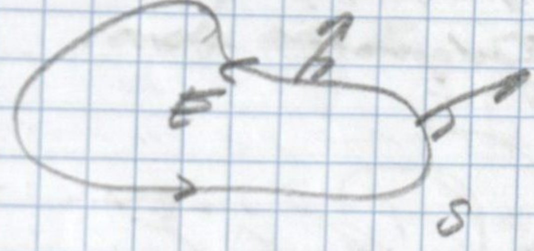
$$\operatorname{div}\vec{F} = \begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}\cdot\begin{bmatrix}x^2\\ xy\\ z\end{bmatrix} \qquad = \ldots = 8\pi.$$

$= 2x + x + 1 = 3x + 1$

Lecture Notes

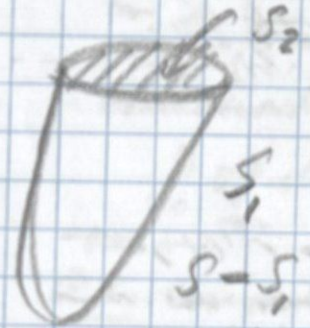
11.29.23

Divergence Theorem:



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

non-closed surfaces:



$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$S = S_1 \cup S_2 = \partial E$

closed surface

Ex. $\vec{F} = \langle x + \tan^{-1} y^3,\ z^3 \ln(x^2+1),\ 0\rangle$

$S_1: x^2+y^2+z = 2 \Rightarrow z = 2 - x^2 - y^2$

$z \geq 1$ Find: upwards orientation of $\iint_{S_1} \vec{F}\cdot d\vec{S}$

$z=1,\ x^2+y^2=1$

$$\therefore \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{1}\,dV - \iint_{S_2} \vec{F}\cdot \underbrace{d\vec{S}}_{\vec{n}\,dS}$$

$$= \iiint_E dV - \underbrace{\iint_{S_2} \begin{bmatrix} x+\tan^{-1}y^3 \\ \ln(x^2+1) \\ 0 \end{bmatrix}\cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} dS}_{0}$$

↳ 0 b/c must be facing outwards from S_2

$$= \int_0^{2\pi}\int_0^1\int_1^{2-r^2} r\,dz\,dr\,d\theta$$

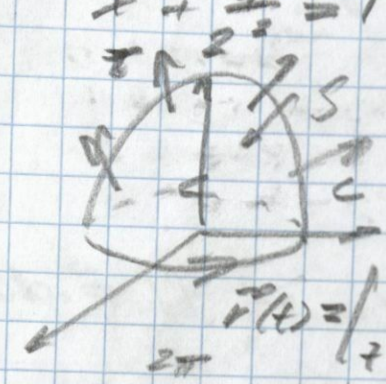
$$= 2\pi\int_0^1 r - r^3\,dr = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}.$$

Calculus III Integral Review Worksheet:

4 options:

1. Fundamental Theorem of Line Integrals
2. Green's Theorem
3. Stokes' Theorem
4. Divergence Theorem

Q1: Let $S: z = 4 - 4x^2 - y^2$, $z \geq 0$ oriented upward. Let $\vec{F} = \begin{bmatrix} x-y \\ x+y \\ ze^{xy} \end{bmatrix}$. Compute $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.



Use Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

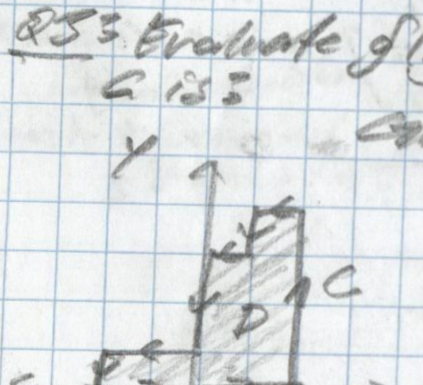
$$= \int_0^{2\pi} \begin{bmatrix} \cos t - 2\sin t \\ \cos t + 2\sin t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ 2\cos t \\ 0 \end{bmatrix} dt$$

$$\vec{r}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ 2\sin t \\ 0 \end{bmatrix}, \quad 0 \leq t \leq \pi$$

$$\Rightarrow \int_0^{2\pi} -\cos t \sin t + \underbrace{2\sin^2 t + 2\cos^2 t}_{2} + 4\cos t \sin t + 0\,dt$$

$$= \int_0^{2\pi} 2 + 3\sin t \cos t\,dt = \int_0^{2\pi} 2 + \tfrac{3}{2}\sin 2t\,dt = 4\pi + 0 = 4\pi.$$

Q5: Evaluate $\oint_C \underbrace{(x^4y^5 - 2y)}_{P}dx + \underbrace{(3x + x^5y^4)}_{Q}dy$ where C is:

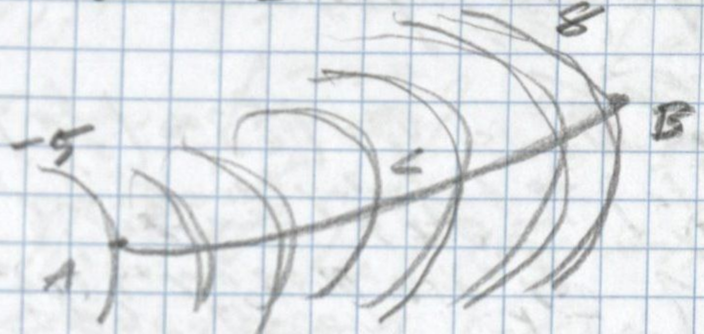


Green's Theorem:

$$\iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{(3+5x^4y^4) - (5x^4y^4 - 2)} dA = \iint_D 5\,dA$$

$$= 5A(D) = 5 \cdot 9 = 45$$

Q 12 - Compute $\int_C \nabla f \cdot d\vec{r}$ where the contour plot of f is given and C is the curve which starts at A and ends at B



Fundamental Theorem of Line Integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 8 - (-5) = 13.$$

Q 7 - Compute $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$, where $\vec{F} = \begin{pmatrix} 4xy + z^2 \\ 2x^2 + 6yz \\ 2xz \end{pmatrix}$

and S ∈ closed in first octant, $x=4$, and $z = 9 - y^2$, with outward orientation.

$$\underset{\text{closed surface}}{\iint_S} \operatorname{curl} \vec{F} \cdot d\vec{S} = 0$$

Stokes Theorem:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S = \emptyset} \vec{F} \cdot d\vec{S} = 0$$

Divergence Theorem:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div} \operatorname{curl} \vec{F}}_{\operatorname{div} \operatorname{curl} \vec{F} \text{ always } = 0} \, dV = 0$$

[Lecture Notes] Integration Review 12.4.23

ex. evaluate $\int_C (x^2+y^2)dx + 2xy\,dy$ where C is parameterized by $\vec{r}(t) = \langle t^3, t^2 \rangle, 0 \le t \le 2$.

FTLI: hope $\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} x^2+y^2 \\ 2xy \end{bmatrix}$ is conservative

test: $Q_x = P_y$ ✓

find potential fn. i.e. $\nabla f = \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$

$\begin{bmatrix} x^2+y^2 \\ 2xy \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} \to f = \int x^2+y^2\,dx = \frac{x^3}{3} + xy^2 + g(y)$

$f_y = 2xy \Rightarrow 0 + 2xy + g'(y) = 2xy$

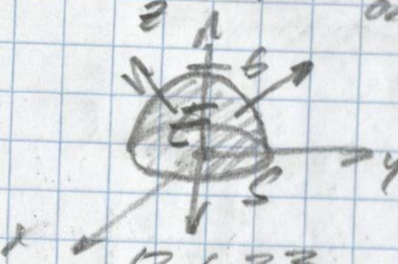
$\therefore f = \frac{x^3}{3} + xy^2$. $\quad g'(y) = 0 \Rightarrow g = 0$

FTLI: $\int_C P\,dx + Q\,dy = f(\vec{r}(2)) - f(\vec{r}(0))$

$= f(8,4) - f(0,0) = \frac{8^3}{3} + 8 \cdot 4^2 - 0$.

ex. Find the flux of $\vec{F} = \langle x, y^2, -2yz \rangle$ across the surface which is the boundary of $z = \sqrt{36-x^2-y^2}$ and $z = 0$, with outwards orientation.

$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1$.



$x^2+y^2+z^2 = 36$

$z \ge 0$

$\Rightarrow 1 \cdot V(E) = \frac{1}{2}\left(\frac{4}{3}\pi \cdot 6^3\right)_0$

divergence theorem: flux $= \iiint_E \operatorname{div} \vec{F}\,dV = \iiint_E 1\,dV$

↳ or use spherical coordinates

12.6.23

[Lecture Notes] Exam 4 Review

S18. Consider $\vec{F} = \begin{bmatrix} e^x \sin z \\ e^y \sin x \\ e^z \sin y \end{bmatrix}$. Find $\operatorname{div} \vec{F}$ and $\operatorname{curl} \vec{F}$

$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} e^x \sin z \\ e^y \sin x \\ e^z \sin y \end{bmatrix} = e^x \sin z + e^y \sin x + e^z \sin y$.

$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} e^x \sin z \\ e^y \sin x \\ e^z \sin y \end{bmatrix} = \begin{bmatrix} e^z \cos y - 0 \\ e^x \cos z - 0 \\ e^y \cos x - 0 \end{bmatrix}$.

$\therefore \vec{F}$ is not conservative/gradient b/c $\operatorname{curl} \vec{F} \ne \vec{0}$.

S19. Find $\int_C y^3\,dx - x^3\,dy$ where C is parameterized by $\vec{r}(t) = \begin{bmatrix} 2\sin t \\ 2\cos t \end{bmatrix}$ from $t=0$ to $t = 2\pi$. (clockwise)

$\int \vec{F} \cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = -\iint_D -3x^2 - 3y^2\,dA$

$= -\int_0^{2\pi}\int_0^2 -3r^3\,dr\,d\theta = -\int_0^{2\pi} \underbrace{\left[-\frac{3}{4}r^4\right]_0^2}_{-12} d\theta = +24\pi$.

negatively oriented

S20. Let S be the surface which is the portion of the cone $x^2+y^2=z^2$ above the xy-plane and below $z=1$, oriented downward. Find the flux $\iint_S \vec F\cdot d\vec S$, where $\vec F=\begin{bmatrix}x\\y\\z^2\end{bmatrix}$

$$\vec S(r,\theta)=\begin{bmatrix}r\cos\theta\\r\sin\theta\\r\end{bmatrix},\quad 0\le r\le 1,\ 0\le\theta\le 2\pi$$

$$\therefore \iint_S \vec F\cdot d\vec S=\iint_S \vec F(\vec S(r,\theta))\cdot(\vec S_r\times\vec S_\theta)\,dS$$

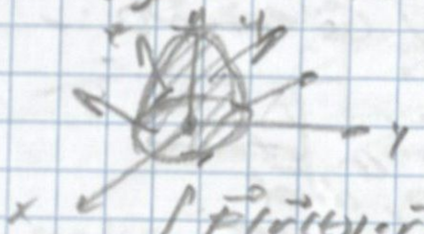
$$\vec S_r\times\vec S_\theta=\begin{bmatrix}\cos\theta\\\sin\theta\\1\end{bmatrix}\times\begin{bmatrix}-r\sin\theta\\r\cos\theta\\0\end{bmatrix}=\begin{bmatrix}-r\cos\theta\\-r\sin\theta\\r\end{bmatrix}\underset{?}{\Rightarrow}\begin{bmatrix}r\cos\theta\\r\sin\theta\\-r\end{bmatrix}$$ oriented downward

$$\therefore \int_0^{2\pi}\int_0^1\begin{bmatrix}r\cos\theta\\r\sin\theta\\r^2\end{bmatrix}\cdot\begin{bmatrix}r\cos\theta\\r\sin\theta\\-r\end{bmatrix}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r^2\cos^2\theta+r^2\sin^2\theta-r^3\,dr\,d\theta=2\pi\left[\frac13 r^3-\frac14 r^4\right]_0^1=\frac{\pi}{6}.$$

S21. Let S be the portion of the paraboloid $z=4-x^2-y^2$ above the xy-plane, oriented upward. Find $\iint_S \operatorname{curl}\vec F\cdot d\vec S$ where $\vec F=\begin{bmatrix}x^2\sin z\\y^2\\xy\end{bmatrix}$

$$\iint_S \operatorname{curl}\vec F\cdot d\vec S=\int_C \vec F\cdot d\vec r$$

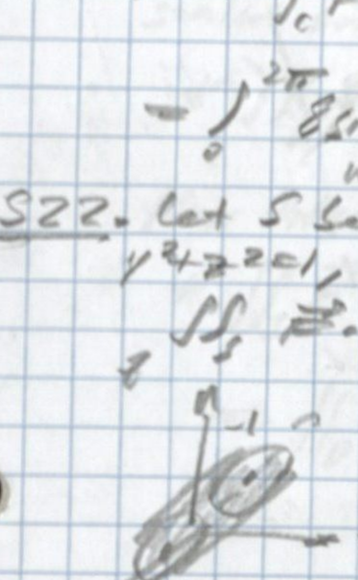


$$\vec r(t)=\begin{bmatrix}2\cos t\\2\sin t\\0\end{bmatrix},\quad \vec r'(t)=\begin{bmatrix}-2\sin t\\2\cos t\\0\end{bmatrix},\quad 0\le t\le 2\pi$$

$$\int_C \vec F(\vec r(t))\cdot\vec r'(t)\,dt=\int_0^{2\pi}\begin{bmatrix}4\cos^2 t\sin 0\\4\sin^2 t\\4\cos t\sin t\end{bmatrix}\cdot\begin{bmatrix}-2\sin t\\2\cos t\\0\end{bmatrix}dt$$

$$=\int_0^{2\pi}8\sin^2 t\cos t\,dt\qquad u=\sin t,\ du=\cos t\,dt\qquad =8\left[\frac{\sin^3 t}{3}\right]_0^{2\pi}=0.$$

S22. Let S be the surface of the solid bounded by $y^2+z^2=1$, $x=-1$, and $x=1$, oriented inward. Find $\iint_S \vec F\cdot d\vec S$ where $\vec F=\begin{bmatrix}3xy^2\\xe^z\\z^3\end{bmatrix}$



$$\operatorname{div}\vec F=\begin{bmatrix}\partial/\partial x\\\partial/\partial y\\\partial/\partial z\end{bmatrix}\cdot\begin{bmatrix}3xy^2\\xe^z\\z^3\end{bmatrix}=3y^2+3z^2$$

$$\Rightarrow -3\int_{-1}^1\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta\,dx=-6\pi\int_{-1}^1\left[\frac14 r^4\right]_0^1dx$$

$$=-\frac{3\pi}{2}\int_{-1}^1dx=-\frac{3\pi}{2}\left[\underbrace{1-(-1)}_{2}\right]=-3\pi.$$

[Independent Notes] Exam 4 Review 12.6.23

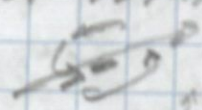
Standard 18: Compute Curl and Divergence of Vector Fields and be able to identify pictures of Vector Fields from an equation

$f: \mathbb{R}^n \to \mathbb{R}$

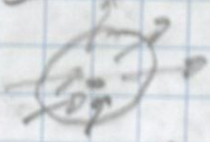
$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

$\underbrace{\vec{\nabla} \times \vec{F}}$ "points along the axis of counterclockwise rotation with length equal to the rate of rotation"

$$* \ \text{curl}\,\vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix}.$$



$$* \ \text{div}\,\vec{F} = \underbrace{\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}}_{\vec{\nabla} \cdot \vec{F}} = P_x + Q_y + R_z.$$

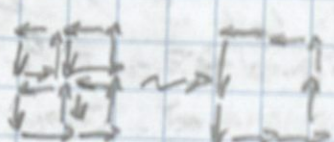


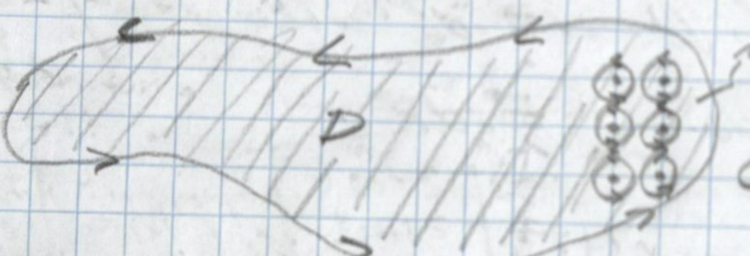
"net fluid escaping an infinitesimal sphere centered at a point"

Standard 19: Use Green's Theorem

"Let C be a positively oriented piecewise smooth, simple closed curve in the plane which bounds a region D. If P and Q have continuous first partials on D, then:

$$* \ \oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$





$C \Leftrightarrow \partial D$

Note: special case of Stokes' Theorem, where line integral is closed, $\vec{n} = \langle 0, 0, 1 \rangle$, and $\vec{F} = \langle P, Q, 0 \rangle$, making $\text{curl}\,\vec{F} \cdot \vec{n} = \begin{vmatrix} 0 \\ 0 \\ Q_x - P_y \end{vmatrix}$.

so, $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\,\vec{F} \cdot d\vec{S}$

$$\Rightarrow \oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

Standard 20: Calculate Surface Integrals of both Scalar (dS) and Flux ($d\vec{S}$)

parametrizing surfaces on region D:

$$* \quad \vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}, \quad (u,v) \in D.$$

ex. parametrize $z = 3\sqrt{x^2+y^2}$:

$$\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ 3\sqrt{u^2+v^2} \end{bmatrix}, \quad (u,v) \in \mathbb{R}.$$

$$\vec{s}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 3r \end{bmatrix}, \quad r \geq 0, \ 0 \leq \theta \leq 2\pi.$$

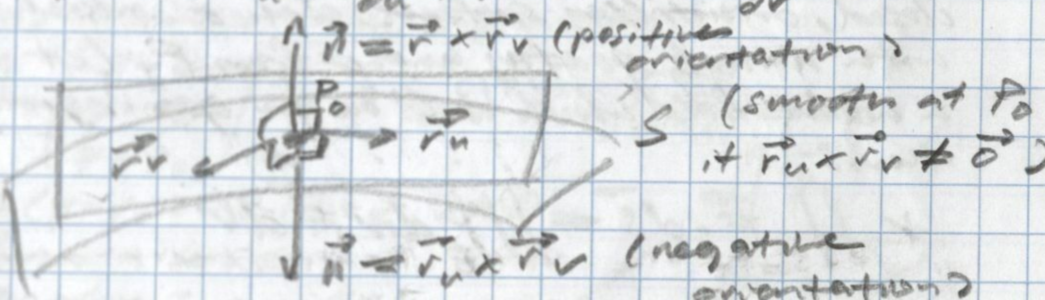
ex. parametrize $x^2+y^2+z^2 = 9$:

$$\vec{q}(\theta,\phi) = \begin{bmatrix} 3\sin\phi\cos\theta \\ 3\sin\phi\cos\theta \\ 3\cos\phi \end{bmatrix}, \quad 0 \leq \phi \leq \pi, \ 0 \leq \theta \leq 2\pi.$$

parametrizing planes and tangent planes to $\vec{r}(u,v)$ surfaces:

$$* \quad \vec{r}(s,t) = \vec{P_0} + s\vec{r}_u + t\vec{r}_v$$

$$\vec{P_0} = \vec{r}(u_0,v_0) \quad \vec{r}_u = \frac{\partial \vec{r}}{\partial u}(u_0,v_0) \quad \vec{r}_v = \frac{\partial \vec{r}}{\partial v}(u_0,v_0)$$



scalar surface integrals: "volume above S"

$$* \quad \iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \|\vec{r}_u \times \vec{r}_v\| \, dA$$

flux integrals: "net rate of flow through S"

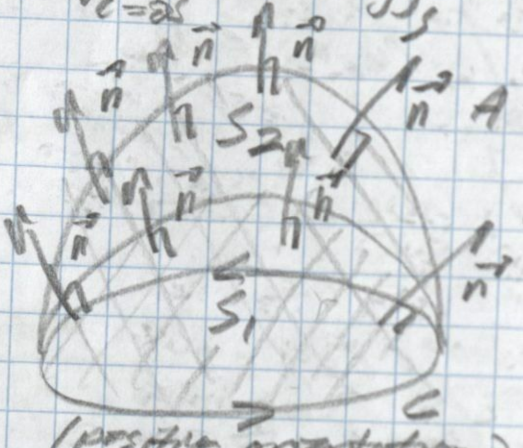
$$* \quad \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dA$$

$$\Leftrightarrow \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \|\vec{r}_u \times \vec{r}_v\| \, dA$$

Standard 21: Use Stokes' Theorem

"Let S be an oriented, piecewise-smooth surface, which is bounded by a simple, closed, piecewise-smooth curve C, and give C the orientation induced by S. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$, which is C^1 on an open region containing S, then:

$$* \int_{C=\partial S} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$$



A trick:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{S_2} \operatorname{curl}\vec{F}\cdot d\vec{S}$$

Swap surfaces to make problem easier...

(positive orientation) $\rightsquigarrow \vec{n} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ ≥ 0

Standard 22: Use Divergence Theorem

"Let E be a bounded solid region in $\mathbb{R}^3$ with boundary S, where S consists of finitely many piecewise-smooth, closed, orientable surfaces, each of which oriented with normals pointing away from E. Let $\vec{F}$ be a vector field which is C^1 on an open region containing E, then:

$$* \iint_{S=\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

"net rate of flow exiting E"

non-closed surfaces:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV - \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$S = S_1 \cup S_2 = \partial E$, now a closed-surface

Independent Notes | Final Exam Review 12.10.23

Standard 01: Use vector operations including dot and cross product

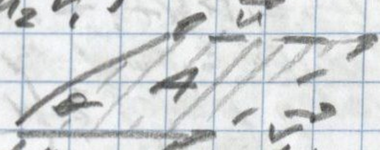
$d^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$

$\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+u_3v_3 = \|\vec{u}\|\|\vec{v}\|\cos\theta$, $\vec{u}\cdot\vec{v}=0 \Leftrightarrow \vec{u}\perp\vec{v}$

$\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$, $\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\hat{u}$, $\text{comp}_{\vec{u}}\vec{v} = \|\text{proj}_{\vec{u}}\vec{v}\|$

$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix} = \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}$, $\vec{u}\times\vec{v}=0 \Leftrightarrow \vec{u}\parallel\vec{v}$

$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta \rightarrow$



$r^2 = (x-h)^2+(y-k)^2+(z-l)^2$

Standard 02: Lines: Find parametrization of a line. Basic Properties, Intersections, Distance

$\vec{r}(t) = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} + t\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{cases}x = x_0+ta\\ y = y_0+tb\\ z = z_0+tc\end{cases}$

symmetric form:

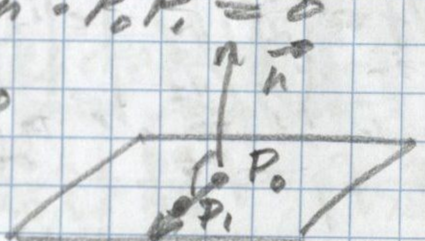
$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

2 lines intersect, are parallel, or are skew

skew lines: non-parallel and non-intersecting

Standard 03: Planes: Find an equation of a plane. Basic Properties, Intersections, Distance

$\vec{r}(t) = \begin{bmatrix}a\\b\\c\end{bmatrix}\cdot\begin{bmatrix}x-x_0\\y-y_0\\z-z_0\end{bmatrix} = 0 \Leftrightarrow \vec{n}\cdot\overrightarrow{P_0P_1} = 0$



$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

2 planes: form a line

1 line and 1 plane: parallel (line contained in plane, line never intersects plane), intersect

$ax+by+cz=d$

Standard 04: Understanding parametrized curves (identifying which plot corresponds to which parametrization; going between equation and parametrization; finding a curve as an intersection of 2 surfaces)

$\vec{r}(t) = \begin{bmatrix}f(t)\\g(t)\\h(t)\end{bmatrix}$, $\lim_{t\to a}\vec{r}(t) = \begin{bmatrix}\lim_{t\to a}f(t)\\\lim_{t\to a}g(t)\\\lim_{t\to a}h(t)\end{bmatrix}$

2 planes intersect at a line

2 surfaces intersect at a curve

$x^2+y^2=r^2 \Leftrightarrow \vec{r}(t) = \begin{bmatrix}r\cos t\\r\sin t\end{bmatrix}$, $0\le t\le 2\pi$

$y^2 = 4ax \Leftrightarrow \vec{r}(t) = \begin{bmatrix}4at\\t\end{bmatrix}$

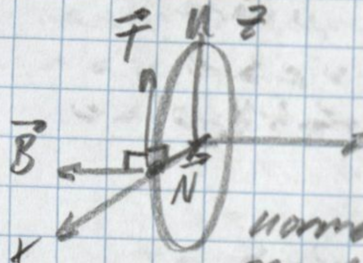
Standard 05: Be able to calculate the derivatives and integrals of space curves and find arc length.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}, \quad \int \vec{r}(t)\,dt = \begin{bmatrix} \int f(t)\,dt \\ \int g(t)\,dt \\ \int h(t)\,dt \end{bmatrix}$$

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t), \quad \|\vec{v}(t)\| = \text{speed}$$

Standard 06: Find the TNB frame, normal plane, and osculating plane, and motion in space



$$\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}, \quad \vec{N} = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

$$\vec{B} = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}, \quad \vec{T} \times \vec{N} \to \vec{B}$$

normal plane: $\perp \vec{T} = \vec{n}$

osculating plane: $\perp \vec{B} = \vec{n}$

$ax + by + cz = d \longrightarrow T, N, B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \vec{r}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

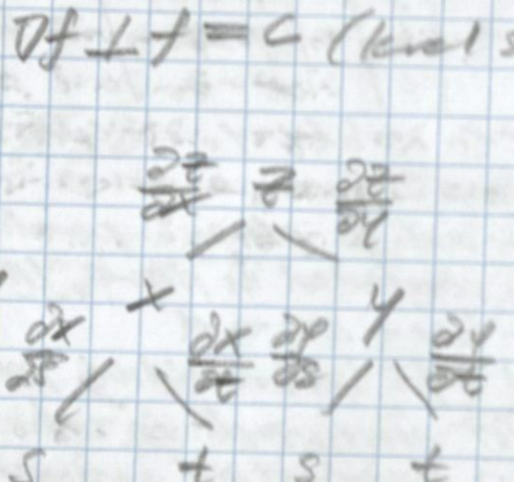
$$\|\vec{a}\| = \sqrt{a_T^2 + a_N^2}, \quad a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{\|\vec{r}\,'(t)\|}, \quad a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$$

Standard 07: Compute partial derivatives including chain rule and implicit differentiation and finding gradient

$$\nabla f = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \quad \nabla f \perp f = c \text{ (level sets)}$$

$$\frac{\partial}{\partial x} fg = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}$$

$$\frac{\partial z}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i}$$



$$f_{xx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

$$\vdots$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

* when computing $\frac{\partial x}{\partial z}$ implicitely, $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial y} = 0$.

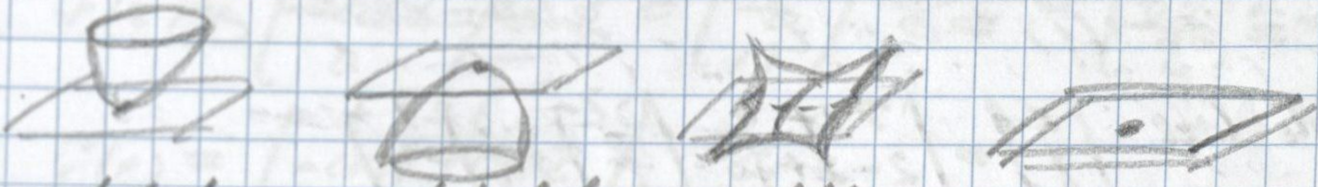
Standard 08: Find a Directional Derivative and find the direction where rate of change is maximized or minimized

$$D_{\hat{u}}f = \nabla f \cdot \hat{u} = \|\nabla f\|\|\hat{u}\|\cos\theta = \|\nabla f\|\cos\theta$$

maximum ROC: $\|\nabla f\|$ in direction of ∇f) $\theta = 0$

minimum ROC: $-\|\nabla f\|$ in direction of $-\nabla f$

Standard 09: Find local extrema



paraboloid up	paraboloid down	saddle	plane
local min	local max	neither	inconclusive
$\det(H_f) > 0$	$\det(H_f) > 0$	$\det(H_f) < 0$	$\det(H_f) = 0$
$f_{xx}, f_{yy} > 0$	$f_{xx}, f_{yy} < 0$		

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}, \quad \det(H_f) = f_{xx}f_{yy} - f_{xy}^2$$

critical points: $\nabla f = 0$

Standard 10: Find absolute extrema using boundary conditions or Lagrange multipliers

Given f is continuous, closed, and bounded on D

1. find critical points ($\nabla f = 0$)
2. parametrize boundary
   a) test critical points ($g_i'(t) = 0$)
   b) test end points (of each segment)

OR use Lagrange Multipliers

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = c \\ h = k \end{cases} \Rightarrow \text{solve for all } x, y, z \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Plug in all points to f and discern extrema

Standard 11: Calculate Double Integrals over rectangular and other simple regions

1. $\vec{a}\cdot\vec{b}=3$, $\|\vec{a}\times\vec{b}\|=3$, find angle between $\vec{a}$ and $\vec{b}$

$\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\theta$ $\quad \|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\theta$

$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}=\frac{3}{3}=1$ $\quad \sin\theta=\frac{\|\vec{a}\times\vec{b}\|}{}=\frac{3}{?}$

$\theta=\cos^{-1}(1)=0.$ E $\quad \sin\theta=\cos\theta$ C

2. the lines $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}7+t\\2-2t\\3t\end{bmatrix}=\begin{bmatrix}7\\2\\0\end{bmatrix}+t\begin{bmatrix}1\\-2\\3\end{bmatrix}$

and $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6+2s\\2-4s\\9+6s\end{bmatrix}=\begin{bmatrix}6\\2\\9\end{bmatrix}+s\begin{bmatrix}2\\-4\\6\end{bmatrix}$

are parallel D

3. equation of plane through $P=(2,2,1)$, $Q=(3,3,1)$, $R=(4,1,4)$

$P_0=\begin{bmatrix}2\\2\\1\end{bmatrix}$, $\vec{PQ}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{PR}=\begin{bmatrix}2\\-1\\3\end{bmatrix}$

$\vec{n}=\vec{PQ}\times\vec{PR}=\begin{bmatrix}1\\1\\0\end{bmatrix}\times\begin{bmatrix}2\\-1\\3\end{bmatrix}=\begin{bmatrix}3-0\\0-3\\-1-2\end{bmatrix}=\begin{bmatrix}3\\-3\\-3\end{bmatrix}\propto\begin{bmatrix}1\\-1\\-1\end{bmatrix}$

$\vec{PQ}\cdot\vec{n}=0$

$(x-1)-(y-1)-z=0$

$x-1-y+1-z=0$

$x-y-z=0$ D

$\vec{PR}\cdot\vec{n}=0$

$(x-2)-(y+1)-(z-3)=0$

$x-2-y-1-z+3=0$

$x-y-z=0$

4. ?

$\vec{r}(t)=\begin{bmatrix}2t\\1+3t\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}+t\begin{bmatrix}2\\3\\0\end{bmatrix}$

) $\frac{x}{4}=\frac{(y-1)}{6}=\frac{z}{2}$ B

c. $\vec{r}(t)=\begin{bmatrix}2\\4\\1\end{bmatrix}-t\begin{bmatrix}2\\3\\0\end{bmatrix}$

5. $\vec{r}'(t) = \begin{bmatrix} 2t \\ \sin t \\ 0 \end{bmatrix}$

$r(t) = \int \vec{r}'(t)\,dt = \begin{bmatrix} t^2 \\ -\cos t \\ 0 \end{bmatrix} + \vec{c}$

$r(2) = \begin{bmatrix} 4 \\ -\cos 2 \\ 0 \end{bmatrix} + \vec{c} = \begin{bmatrix} 5 \\ 0 \\ 17 \end{bmatrix} \Rightarrow \vec{c} = \begin{bmatrix} 1 \\ \cos 2 \\ 17 \end{bmatrix}$

$\vec{r}(t) = \begin{bmatrix} t^2+1 \\ -\cos t + \cos 2 \\ 0 + 17 \end{bmatrix}$ C

6. $\vec{r}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$. Normal plane at $t=1$, $\vec{r}(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{1+4t^2+9t^4}} \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \xRightarrow{t=1} \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\vec{n} = \perp \vec{T}$:

$(x-1)+2(y-1)+3(z-1)=0$

$x-1+2y-2+3z-3=0$

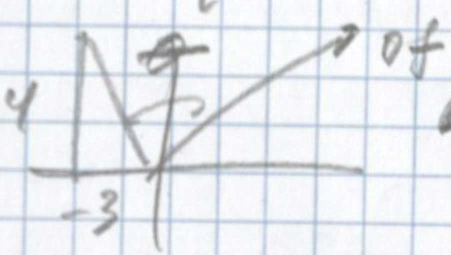
$x+2y+3z=6$ B

7. $f(x,y,z) = x^3yz^5$, find f_{xyyz} E

$f_x = 3x^2yz^5$. $f_{xy} = 3x^2z^5$. $f_{xyy} = 0$. $f_{xyyz} = 0$.

8. find $D_{\hat{u}} f$ with $f(x,y) = xy$ at $(2,1)$ with $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

$\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{u} = \frac{1}{\sqrt{9+16}} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$



$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \|\nabla f\| \|\hat{u}\| \cos\theta$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \frac{1}{5}$. C

9. $f(x,y) = x^4+y^4$

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad D$$

10. abs max of $f(x,y,z) = y^2 - 4z$ subject to $x^2+y^2+z^2 = 9$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 9 \end{cases}, \quad \nabla f = \begin{pmatrix} 0 \\ 2y \\ -4 \end{pmatrix}, \quad \nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\text{sys} \begin{cases} ① \ 0 = 2\lambda x \rightsquigarrow x = 0 \text{ or } \lambda = 0 \\ ② \ 2y = 2\lambda y \rightsquigarrow y = 0, \ y = 1 \quad y = 0 \\ ③ \ -4 = 2\lambda z \quad z^2 = 9 \quad z^2 = 8 \quad ①, ② \text{ don't work} \\ ④ \ x^2+y^2+z^2 = 9 \quad z = \pm 3 \quad z = \pm\sqrt{8} \end{cases}$$

⑤

③ $2\lambda = -\frac{4}{z} \Rightarrow \lambda = -\frac{2}{z} \rightsquigarrow z = 2\left(-\frac{2}{z}\right) = -\frac{4}{z}$

③ $\lambda = -\frac{2}{-2} = 1$ $\qquad 2z = -4$

$\qquad z = -2$

② $2y = 2y \Rightarrow y = 0$

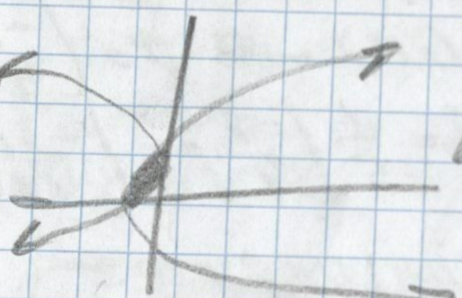
① $x = 0$

A

④ $0 + y^2 + 4 = 9$

$y^2 = 5 \Rightarrow f = 13$

11. Area between $x = y^2 - 1$ and $x = -(y-1)^2$



$$\int_0^1 \int_{y^2-1}^{-(y-1)^2} dx\, dy \quad B$$

12. $\iint_D x^2+y^2\,dA$, D $r=2$ at $(0,0)$

$$\iint_0 r^2\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

$$= 2\pi\left[\frac{1}{4}r^4\right]_0^2 = 2\pi[4] = 8\pi.$$ C

13. $$\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\int_{z=0}^{z=1-x^2-y^2} x\,dz\,dy\,dx$$

to $dy\,dx\,dz$

$$z = 1-x^2-y^2$$
$$y^2 = 1-x^2-z$$
$$y = \pm\sqrt{1-x^2-z}$$

$$y = \pm\sqrt{1-x^2}$$
$$y^2 = 1-x^2$$
$$x^2 = 1-y^2$$
$$x = \pm\sqrt{1-y^2} =$$

$$\int_0\int_{-1}^{1}\int_{-\sqrt{1-x^2-z}}^{\sqrt{1-x^2-z}} x\,dy\,dx\,dz$$ E?

14. region above cone $z = 2\sqrt{x^2+y^2}$ and below $z=6$

$$\int_0^{2\pi}\int\int_{2r}^{6} r\,dz\,dr\,d\theta$$ C?

15. area of $4x^2+9y^2 \le 36$ using $x = 3u\cos v$, $y = 2u\sin v$

$4(9u^2\cos^2 v) + 9(4u^2\sin^2 v) \le 36$

$36u^2 \le 36 \Rightarrow u^2 \le 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3\cos v & -3u\sin v \\ 2\sin v & 2u\cos v \end{vmatrix}$$

$$= 6u\cos^2 v + 6u\sin^2 v = 6u \qquad B$$

$$\int_0^{2\pi}\int_0^1 6u^3\,du\,dv = 2\pi\left(\frac{3}{2}u^4\right)\Big|_0^1 = \frac{6\pi}{2} = 3\pi.$$

16. arc length of $x^2+4y^2=4$ from $(2,0)$ to $(-\sqrt{2}, \frac{\sqrt{2}}{2})$

$\frac{x^2}{4} + y^2 = 1$

$\vec{r}(t) = \begin{bmatrix} 2\cos t \\ \sin t \end{bmatrix}, \ 0 \le t \le \frac{3\pi}{4}$ E ?

17.

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} x\sin^2 z \\ y\cos^2 z \\ -z \end{bmatrix} = \sin^2 z + \cos^2 z - 1 = 0. \quad B$$

18.

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\vec{F}(x,y) = \frac{\langle -x, -y \rangle}{(x^2+y^2)^{3/2}}$ ↳ $(1,0)$ ↳ $(3, -\sqrt{7})$

$\vec{F} = \begin{bmatrix} -x \\ -y \end{bmatrix} \Rightarrow f = \int -x\,dx = -\frac{x^2}{2} + g(y)$

$\Rightarrow f_y = -y = g'(y) \Rightarrow g(y) = \int -y\,dy$

$\therefore f = -\frac{x^2}{2} - \frac{y^2}{2}$

$f(3, -\sqrt{7}) = -\frac{9}{2} - \frac{7}{2} = -8$ A ?

$f(1,0) = -\frac{1}{2} - 0 = -\frac{1}{2}$

$f(\vec{r}(b)) - f(\vec{r}(a)) = 0$ $\frac{15}{2}$

19. $\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\vec{F} = \begin{bmatrix} 2y + ye^x \\ 2x + e^x \end{bmatrix} \begin{matrix} P \\ Q \end{matrix}$

$\iint_D dA = 2$, $\iint_D (7 + e^x - 2 - e^x)\, dA$

$\Rightarrow \iint_D 5\, dA = 5 \iint_D dA = 5(2) = 10.$ C

20. $\iint_S \vec{F}\cdot d\vec{S}$, $\vec{F} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$

$S: z = x^2 - y^2$, $-1 \le x \le 1 \Rightarrow -1 \le u \le 1$, $-1 \le y \le 1 \Rightarrow -1 \le v \le 1$

$\vec{r}(u,v) = \begin{bmatrix} u \\ v \\ u^2 - v^2 \end{bmatrix}$

$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} 1 \\ 0 \\ 2u \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix} = \begin{bmatrix} 0 - 2u \\ 0 + 2v \\ 1 \end{bmatrix} = \begin{bmatrix} -2u \\ 2v \\ 1 \end{bmatrix}$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dA$ B

$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2u \\ 2v \\ 1 \end{bmatrix} dA = \int_{-1}^{1}\int_{-1}^{1} -2u^2\, du\, dv$

21.
C: $\vec{r}(t) = \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} \Rightarrow x^2+y^2=1$

surface S with negative orientation
(wrong)

C

22. $\iint_S \langle \frac{x^3}{3}, \frac{y^3}{3}, \frac{xy}{2} \rangle \cdot d\vec{S}$

S: between $z=1$ and $z=5$, and $x^2+y^2=1$
with outwards orientation

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$

$\operatorname{div} \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} x^3/3 \\ y^3/3 \\ xy/2 \end{bmatrix} = x^2+y^2+0$

$\iiint_E r^2 r \, dr \, d\theta \, dz = \int_1^5 \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta \, dz$

$2\pi \int_1^5 \left[\frac{1}{4} r^4\right]_0^1 dz = 2\pi \int_1^5 \frac{1}{4} dz$

$= 2\pi \left[\frac{z}{4}\right]_1^5 = 2\pi \left[\frac{5}{4} - \frac{1}{4}\right] = 2\pi.$

E