Linear Algebra & Diff Eqn Lecture 1: 1.17.24

Systems of Linear Equations:

Linear Equation: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

$a_1, \dots a_n$ are coefficients from $\mathbb{R}$ or $\mathbb{C}$

$x_1, \dots x_n$ are variables

b is a constant term ($\mathbb{R}$ or $\mathbb{C}$)

* A system of linear equations is a collection of m linear eqn in n variables

* Goal is to discover/find/describe solution set to a system of linear equations

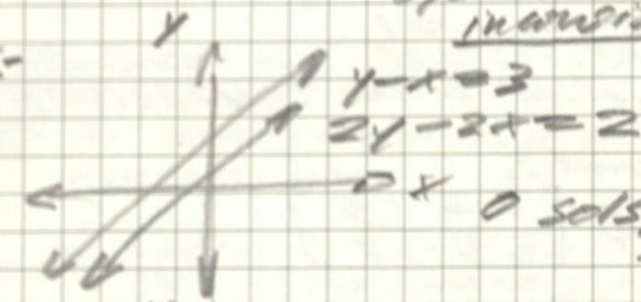
ex. $2x - 3y = 1$ sol. $x = 2,\ y = 1$.
$x + 4y = 6$

Nonsol. $x = 1,\ y = 2$

* Linear systems may have $\overset{\uparrow}{0}$ or $\underbrace{1 \text{ or } \infty}$ sols

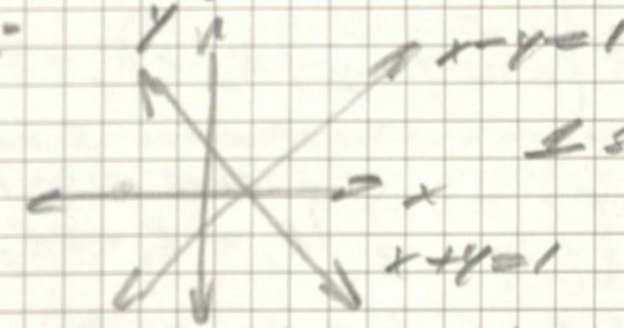
system is inconsistent ; system is consistent

ex.



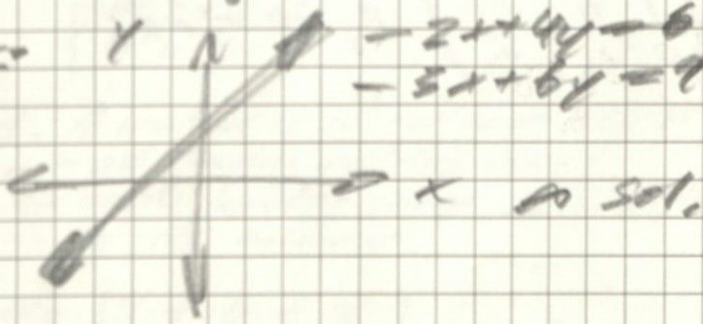
0 sols, inconsistent system.

ex.



1 sol. consistent system.

ex.



∞ sol. consistent.

* linear equation in $\mathbb{R}^3$ are planes: $ax + by + cz = d$

Matrices:

* A $p\times q$-matrix is a rectangular array of "p by q" numbers with p rows and q columns

We can represent linear systems in matrix form as follows:

Ex. $(*)\begin{cases} 2x-3y=1 \\ x+4y=6 \end{cases} \Longleftrightarrow \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 1 & 4 & 6 \end{array}\right]$

Augmented matrix of linear system $(*)$ 2×3

$\Longleftrightarrow \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$

coefficient matrix of $(*)$

Solving linear systems:

ex. $\begin{cases} x_1+3x_2-16x_3=0 \longrightarrow x_1=1 \\ x_2-4x_3=1 \longrightarrow x_2=5 \\ 2x_3=2 \rightarrow x_3=1 \end{cases}$

* triangular form, solved with "backsolving"

How do we solve in general?

ex. $\begin{cases} x+2y=3 \\ 3x+5y=7 \end{cases} \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 5 & 7 \end{array}\right]$

$\Rightarrow \begin{cases} x+2y=3 \\ 0-y=-2 \end{cases} \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & -2 \end{array}\right]$ $R_2 \to R_2-3R_1$

"replace row 2 with −3 times row 1"

$\Rightarrow \begin{cases} x+0=-1 \\ 0-y=-2 \end{cases}$

$\Longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & -2 \end{array}\right]$ $R_1 \to R_1+2R_2$

$\Rightarrow \begin{cases} x=-1 \\ y=2 \end{cases} \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$ $R_2 \to -1\cdot R_2$

Elementary Row Operations:

* To solve linear systems, we can use Elementary Row Operations (EROs) to eliminate variables

EROs: (I) Replace a row with itself + (constant) another row
$R_i \to R_i + cR_j \quad i \neq j$ (replacement)

(II) Swap two rows
$R_i \leftrightarrow R_j \quad i \neq j$ (interchange)

(III) Multiply a row by a nonzero constant
$R_i \to cR_i \quad c \neq 0$ (scaling)

* Def. Two matrices are called Row Equivalent if they can be transformed into one another by performing a sequence of EROs

* Facts. - Row operations are reversable
- If the augmented matrices of two linear systems are equivalent, then those linear systems have the same solution set

Row Reduction and Eschelon Forms:

Def. The leading entry of a nonzero row is the leftmost nonzero entry

A matrix is in Row Eschelon Form (REF) if both:
(I) all nonzero rows are above zero rows
(II) each leading entry is to the left of all leading entries of lower rows

Key. $(\cdot)$ - leading entry / pivot
$(*)$ - something possibly nonzero

ex. $\begin{bmatrix} \cdot & * & * & * \\ 0 & \cdot & * & * \\ 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix}$ in REF

Def. A matrix is in reduced REF if it is in REF and:
- all leading entries are 1
- each leading entry is the only nonzero entry in its column

ex. $\begin{bmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \end{bmatrix}$

* Pivot rows and columns are columns and rows with pivot entries

Lecture 2: 1.19.24

Recap:

$$\begin{matrix} 2x - 3y = 1 \\ x + 4y = 6 \end{matrix} \sim \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 1 & 4 & 6 \end{array}\right]$$

EROs:

(I) replacement $R_i \to R_i + cR_j$ $c \neq 0$.

(II) interchange $R_i \leftrightarrow R_j$

(III) scaling $R_i \to cR_i$ $c \neq 0$

REF: (generalizes triangular form)

- All nonzero rows must be above zero rows
- Every leading term of a row must be to the left of lower leading terms

ex: $\left[\begin{array}{ccc|c} 2 & 5 & 4 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

RREF: in addition to REF:

- all leading entries in rows are 1
- everything above + below leading entries is 0

ex: $\left[\begin{array}{ccc|c} 1 & 6 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

* pivot rows and columns are rows and columns with pivot entries

* free rows and columns are rows and columns without pivot entries

Systems of Equations (II):

Fact: Any matrix can be row reduced to REF by using EROs.

Note: REF is not necessarily unique

Procedure to Convert Matrix into REF:

1. Find the leftmost nonzero column. This will be a pivot column

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 7 \end{bmatrix}$$

2. Interchange rows using ERO (II) to get a nonzero entry of this leftmost column to the top

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 2 & 0 & 1 & 7 \end{bmatrix}.$$

3. Add multiples of the top row to lower rows using ERO (I) to get 0s below leading entry

$$R_3 \to R_3 - 2R_1 \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix}$$

4. The top row is now finished. Cover it up and repeat steps 1-3 for lower rows

$$R_3 \to R_3 + R_2 \begin{bmatrix} 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Fact</u>: One can reduce any matrix to RREF and the result will be unique

<u>Procedure</u> to reduce a matrix from REF to RREF: (in terms of rows)

... assume in REF ...

5) Add multiples of the lowest pivot entry to higher rows using ERO (I) to get 0s above the pivot entries

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim R_2 \to R_2 + 3R_3 \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\sim R_1 \to R_1 + 5R_3 \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim R_1 \to -2R_2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6) Scale all pivot entries to 1.

$$\sim R_1 \to \tfrac{1}{2}R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim R_2 \to \tfrac{1}{2}R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\sim R_3 \to -1R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Pivots</u>: a leading entry of a nonzero row once in REF or RREF

- the number of pivot entries = number of pivot rows = number of pivot columns
- call this number the <u>rank</u> of a matrix

<u>Procedure</u> for solving linear systems of equations:

1) Write augmented matrix of the system and put it into REF
2) Check for inconsistency. The system is inconsistent (number of soln is 0) if there is any row of the form:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & c \end{array}\right] \quad c \neq 0. \text{ otherwise consistent}$$

3) Assume now the system is consistent. Variables corresponding to pivot columns are called <u>bound</u> variables. Non pivot columns are called free columns and their variables are called <u>free</u>
4) Put the matrix into RREF and write the corresponding system of equations. Rewrite system of eqn (i.e. solve) so that free variables are on the right hand side (RHS) of = and the bound variables are on the LHS of =
5) Choose a parameter for free variable (r, s, t, etc) and rewrite all vars in terms of parameters

ex: Find for which constants a the linear system is consistent and find the solution set for each consistent a.

$$\begin{aligned}
x + 0y + 2z + u - v + 2w &= -1\\
2x + 0y + 4z + 2u - 2v + 4w &= -2\\
2x + 0y + 4z + 3u - 3v + 7w &= -3\\
-x + 0y - 2z - 2u + 2v - 4w &= 1\\
3x + 0y + 6z + 5u - 5v + 13w &= a-7
\end{aligned}$$

$$\sim \left[\begin{array}{cccccc|c}
1 & 0 & 2 & 1 & -1 & 2 & -1\\
2 & 0 & 4 & 2 & -2 & 4 & -2\\
2 & 0 & 4 & 3 & -3 & 7 & -3\\
-1 & 0 & -2 & -2 & 2 & -4 & 1\\
3 & 0 & 6 & 5 & -5 & 13 & a-7
\end{array}\right]$$

$$\sim \left[\begin{array}{cccccc|c}
1 & 0 & 2 & 1 & -1 & 2 & -1\\
0 & 0 & 0 & 1 & -1 & 3 & -1\\
0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & a-1\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

This system is consistent exactly when $a-1=0$ which is when $a=1$

$$\sim \left[\begin{array}{cccccc|c}
1 & 0 & 2 & 1 & -1 & 2 & -1\\
0 & 0 & 0 & 1 & -1 & 3 & -1\\
0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$\sim \left[\begin{array}{cccccc|c}
1 & 0 & 2 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 1 & -1 & 0 & 2\\
0 & 0 & 0 & 0 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

rewrite:
$$\begin{aligned}
x + 2z &= -1\\
\sim\quad u + v &= 2\\
w &= -1
\end{aligned}$$

rewrite:
$$\begin{aligned}
x &= -1 - 2z\\
\sim\quad u &= 2 - v\\
w &= -1
\end{aligned}$$

choose parameters of free variables:

$y = r$
$z = s$
$v = t$

parametrize form of solution:

parametric vector form:

$x = -1 - 2s$
$y = r$
$z = s$
$u = 2 - t$
$v = t$
$w = -1$

$$\sim \begin{bmatrix} x\\ y\\ z\\ u\\ v\\ w\end{bmatrix} = \begin{bmatrix} -1-2s\\ r\\ s\\ 2-t\\ t\\ -1\end{bmatrix} = \begin{bmatrix} -1\\ 0\\ 0\\ 2\\ 0\\ -1\end{bmatrix} + r\begin{bmatrix} 0\\ 1\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + s\begin{bmatrix} -2\\ 0\\ 1\\ 0\\ 0\\ 0\end{bmatrix} + t\begin{bmatrix} 0\\ 0\\ 0\\ -1\\ 1\\ 0\end{bmatrix}$$

Lecture 3: 1.22.24

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Linear Systems:

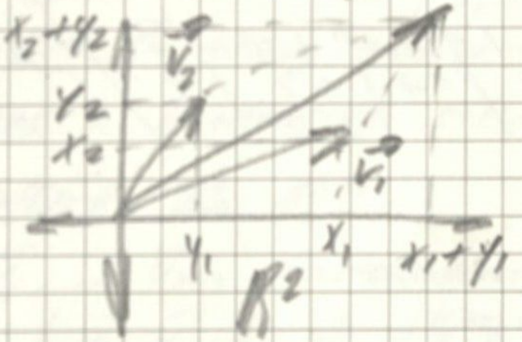
$$\begin{cases} a_{11}x_1 + \dots a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots a_{mn}x_n = b_m \end{cases}$$

Augmented Matrix:

$$[A \,\vdots\, b] \in A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Row Reduction (Gauss Elimination):

- reduce $[A \,\vdots\, b]$ to RREF
- can find a solution or show that it does not exist
- looks at linear systems from a "vector perspective"



$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_2 \end{bmatrix}$$

$$c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

Vector in $\mathbb{R}^n$ are:

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Def. the operation described is called a linear combination of vectors

define. $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$

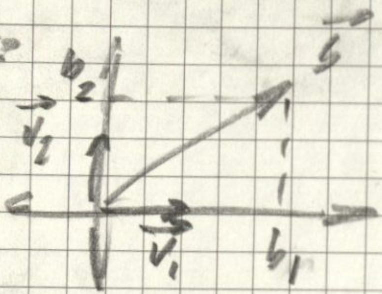
For m vectors in $\mathbb{R}^n$:

$\vec{v}_1, \dots \vec{v}_m$ and m scalars $c_1 \dots c_m$

$$\begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \\ \vdots \\ x_{nm} \end{bmatrix} \rightsquigarrow c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m \rightsquigarrow \begin{bmatrix} c_1x_{11} + \dots c_m x_{1m} \\ \vdots \\ c_1 x_{n1} + \dots + c_m x_{nm} \end{bmatrix}$$

Question. Fix vectors $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{b}$. When $\vec{b}$ a linear combination of $\vec{v}_1, \dots, \vec{v}_m$?

Ex.



$\vec{b} = b_1 \vec{v}_1 + b_2 \vec{v}_2$

where $\vec{v}_1, \vec{v}_2 \in \hat{i}, \hat{j}$

Ex.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}; \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

ask: are there c_1, c_2 where

$$c_1 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Aug Matrix:

$$\begin{bmatrix} -c_1 + c_2 \\ 4c_1 - 2c_2 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 4 & -1 & 2 \\ 2 & -1 & 1 \end{array}\right]$$

Gauss:

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \to c_1 = 2, \; c_2 = 3$$

$$2\vec{v}_1 + 3\vec{v}_2 = \vec{b}$$

Thm. System with the augmented matrix

$$[A \mid b] = [\underbrace{\vec{v}_1 \mid \vec{v}_2 \dots \mid \vec{v}_m}_{A} \mid \vec{b}]$$

is consistent if and only if $\vec{b}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$

Def. A collection (set) of all possible linear combinations of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called their span.

$$\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) = \{ c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_m \vec{v}_m \mid c_1, \dots, c_m \in \mathbb{R} \}$$

If: $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) = \mathbb{R}^n$

then $\vec{v}_1, \dots, \vec{v}_m$ is called a spanning set

Def. Vectors $\vec{v}_1, \dots \vec{v}_m$ in $\mathbb{R}^n$ are dependent if there are scalars $c_1, \dots c_m$ not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$

$\vec{v}_1, \dots \vec{v}_m$ are linearly independent if such $c_1, \dots c_m$ do not exist

Ex.1 $\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}0\\1\\-1\end{bmatrix}}_{\vec{v}_3}$.

$\vec{v}_1 + \vec{v}_3 = \vec{v}_2 \Rightarrow 1\cdot\vec{v}_1 - 1\cdot\vec{v}_2 + 1\cdot\vec{v}_3 = \vec{0}$. lin dependent.

Ex.2

$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

if $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix}c_1+c_2\\c_1-c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2c_1\\2c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow c_1 = 0, c_2 = 0$. $\vec{v}_1, \vec{v}_2$ are independent

Rephrase in terms of linear systems:

$\vec{v}_1, \dots \vec{v}_m$ are dependent if the systems with the augmented matrix $\left[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_m \,\middle|\, \begin{matrix}0\\\vdots\\0\end{matrix}\right]$

(if $\vec{b}$ in any matrix $= \vec{0}$, matrix is homogeneous)

has a nontrivial solution

Corollary. In $\mathbb{R}^n$

$\vec{v}_1, \dots \vec{v}_m$ $(m > n)$, they are always lin dependent

$\left[\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4 \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right]$

RREF $\left[\begin{matrix}1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right]$ $\Rightarrow$ there is a solution

$\Rightarrow$ linear dependence

Simplest linear equations:

$ax = b \Rightarrow x = a^{-1}b$

? $\Rightarrow A \overset{?}{\cdot} \vec{x} = \vec{b}$?

Lecture 4: 1.24.24

Write linear systems in a way similar to the simplest possible equations $a \cdot x = b$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_n \end{cases} \rightarrow \begin{bmatrix} a_{11}x_1 + \dots a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots a_{mn}x_n \end{bmatrix} = \vec{b}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Define: $\underset{m \times n}{A} \cdot \underset{n\text{-comp}}{\vec{x}}$ = left-hand side of our linear system

(row) x (columns)

Recall: $m \times n$ matrix $\hookrightarrow$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$$

1st column ... 1st row ...

Addition of Matrices:

If A and B are $m \times n$ matrices

$A = (a_{ij})_{i=1}^{m}{}_{j=1}^{n}$

$B = (b_{ij})_{i=1}^{m}{}_{j=1}^{n}$

Define: $A + B = (a_{ij} + b_{ij})_{i=1}^{m}{}_{j=1}^{n}$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}\,b_{1n} \\ \vdots \\ a_{m1}+b_{m1} & a_{m2}\,b_{m2} & \dots & a_{mn}\,b_{mn} \end{bmatrix}$$

Ex:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & 6 \end{bmatrix}$$

Multiplication of Scalars: $c \cdot A = (ca_{ij})_{i=1}^{m}{}_{j=1}^{n}$

Properties: $A + B = B + A$

$$0 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots \\ 0 & \dots & 0 \end{bmatrix} \Rightarrow A + 0 = 0 + A;\ A + (-1)A = 0$$

Multiplication of Matrices:

A — $m \times n$ matrix
B — $n \times p$ matrix

want to define $A \cdot B \overset{?}{=}$ $m \times p$ matrix

we already have definition for a product of $m \times n$ and $n \times 1$ matrices

$$A \cdot B = A \cdot [\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_p]$$

def.

$$\Rightarrow [A\vec{b}_1 \mid A\vec{b}_2 \mid \cdots \mid A\vec{b}_p] \quad \begin{array}{l} p \text{ columns} \\ m \text{ rows}\end{array}$$

Definition $A \cdot B$ is an $m \times p$ matrix such that its entry in the ith row and jth column is

$$(A \cdot B)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$= [a_{i1} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\rightarrow A \cdot \vec{x} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$

$$A \cdot B = [a_1 \cdots a_n] \cdot \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \underset{m\times 1}{a_1} \underset{1\times p}{B_1} + a_2 B_2 + \cdots + a_n B_n$$

Ex. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 6 & 2 \end{bmatrix}$

Properties.

? Is $\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{n\times p}{B} \cdot \underset{m\times n}{A}$

A. unless $p = m$, the right-hand side is not even defined!

? What if A and B are square matrices of the same size

$$\underset{m\times m}{A} \cdot \underset{m\times m}{B} \overset{?}{=} B \cdot A$$

A. generally, No!

Ex. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$A \cdot B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \neq B \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Associativity:

$\underset{m\times n}{A} \quad \underset{n\times p}{B} \quad \underset{p\times q}{C}$

$$\underbrace{\underset{m\times p}{(A\cdot B)}\ \underset{p\times q}{C}}_{m\times q} = \underbrace{\underset{m\times n}{A}\ \underbrace{(B\cdot C)}_{n\times q}}_{m\times q}$$

Assume: $m=n$ & A is a square matrix

- diagonal $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & 0\ d_n \end{bmatrix}$, e.j. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

- $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \to \underset{n\times n}{A}\cdot I = A$

(Identity Matrix) $\quad I\cdot A = A$

If A is square, we can define:

$A^2 = A\cdot A$

$A^3 = (A^2)\cdot A$

$\cdots$

$A^k = \underbrace{A\cdot A\cdot \ldots \cdot A}_{k \text{ times}}$

Ex. 1) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = A\cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = (A^2)\cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

... nilpotent matrices

Ex. 2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\vdots$

$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Ex. 3)

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$,

$A^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$, $A^5 = \begin{bmatrix} 8 & \cdot \\ \cdot & \cdot \end{bmatrix}$

... fibonacci numbers

Lecture 5:

1.23.24

$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \Rightarrow$ An $m\times p$ matrix whose i,j entry is $=$

$a_{i1}b_{1j} + a_{i2} + b_{2j} + \dots + a_{in} + b_{nj}$

$= \begin{bmatrix} a_{i1} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$

1) $A \cdot B \neq B \cdot A$

2) $(\underset{m\times n}{A} \cdot \underset{n\times p}{B}) \cdot \underset{p\times q}{C} = A \cdot (C \cdot B)$

3) Let A be square $n\times n$ and

$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \Rightarrow A \cdot I = I \cdot A = A$

Recall. For usual multiplication:

$a \cdot \frac{1}{a} = a \cdot a^{-1} = 1$

Q. Are there inverses for $n\times n$ matrices?

B is an inverse of A if: $A \cdot B = B \cdot A = I$

Notation: $B = A^{-1}$

Ex: 1) $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ Suppose it has an inverse B

$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\underbrace{\underbrace{B \cdot A}_{I} \cdot \underbrace{A \cdot B}_{I}}_{I} = B \cdot \underset{0}{A^2} \cdot B = 0$

contradiction.

not all matrices are invertible

2) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad A \cdot B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \frac{1}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I.$

$A \cdot A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

has an inverse.

$B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ sidenote. where $ad-bc \neq 0$

Also, if $A\vec{x} = \vec{b}$, $\quad$ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A^{-1}\vec{b} = \vec{x}$! $\quad$ also $[A \mid I] \xrightarrow{EROs} [I \mid A^{-1}]$

Linear Transformations:

Transformation T takes a vector from $\mathbb{R}^n$ and produces another vector $\mathbb{R}^m$

$$T: \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{x} \dashrightarrow T(\vec{x})$$

Def: A linear transformation

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

satisfies 2 properties:

1) for any $\vec{u}, \vec{v}$ in $\mathbb{R}^n$
$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

2) for any $\vec{v}$ in $\mathbb{R}^n$ and scalar c
$$T(c\vec{v}) = cT(\vec{v})$$

$$\Rightarrow T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \dots + c_kT(\vec{v}_k)$$

Ex: 1) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y+z \\ y-2z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = T\left(\begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix}\right) = \begin{bmatrix} (x+x')-(y+y')+(z+z') \\ (y+y')-2(z+z') \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} (x-y+z) \\ (y-2z) \end{bmatrix}}_{T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)} + \underbrace{\begin{bmatrix} (x'-y'+z') \\ (y'-2z') \end{bmatrix}}_{T\left(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right)}$$

3) Let A be an $m \times n$ matrix and define

$$T_A(\vec{x}) \overset{def}{=} A \cdot \vec{x} \qquad T_A: \mathbb{R}^n \to \mathbb{R}^m$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

linear

why? $T_A(\vec{x}+\vec{y}) = A\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix} = A\vec{x}+A\vec{y}$

Conclusion: any $m \times n$ matrix leads to a linear transformation

Q: vice versa? YES!

$$= \begin{bmatrix} a_{11}(x_1+y_1) + \dots a_{1n}(x_n+y_n) \\ \vdots \end{bmatrix}$$

For this, need special vectors in $\mathbb{R}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + \dots a_{1n}x_n \end{bmatrix} + \begin{bmatrix} a_{11}y_1 + \dots a_{1n}y_n \end{bmatrix}$$

For any $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ we can write:

$$\vec{x} = x_1 \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \dots + x_n \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots x_n\vec{e}_n$$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation:

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n)$$

Linear:

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \dots + x_n T(\vec{e}_n)$$

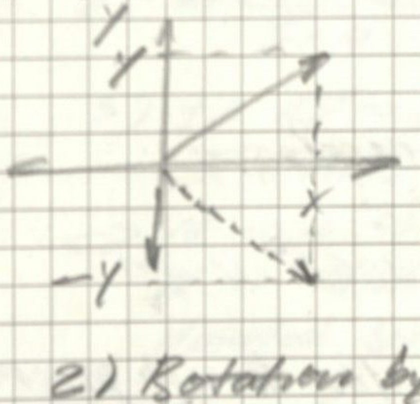
$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_n) \end{bmatrix}}_{A} \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = A\vec{x}$$

Fact: Any linear transformation T can be written as T_A for some matrix A, called the standard matrix representation of T

Notation: $[T]$

Sidenote: A^T is a "transposed" matrix where rows and columns are flipped along diagonal

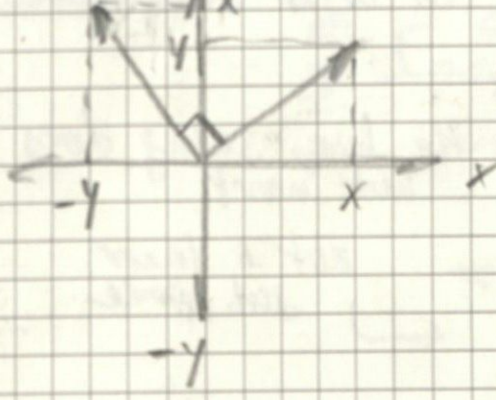
Ex: 1) Reflection



$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\-y\end{bmatrix} \qquad \begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}^T = \begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix}$$

$$= x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

2) Rotation by $90°$ or $\frac{\pi}{2}$

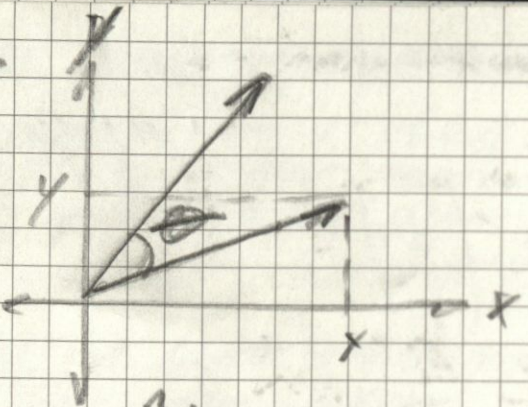


$$-y \cdot x + x \cdot y = 0$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}-y\\x\end{bmatrix}$$

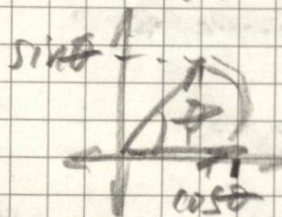
$$= x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Ex:

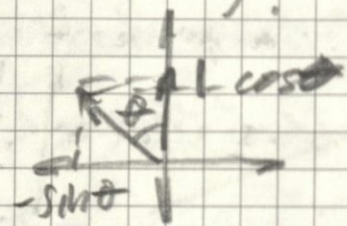


$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$\hat{i}$!



$\hat{j}$!



Lecture 6: 1.29.24

Notion: Dimension, Rank

• Subspaces (linear)

$\mathbb{R}^n$: 1) $\vec{0} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$ in $\mathbb{R}^n$

2) $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ we can consider $\vec{u}+\vec{v}$

3) any vector $\vec{u}$ in $\mathbb{R}^n$ can be multiplied by a scalar

Def. S a subset of $\mathbb{R}^n$ is called a linear subspace if

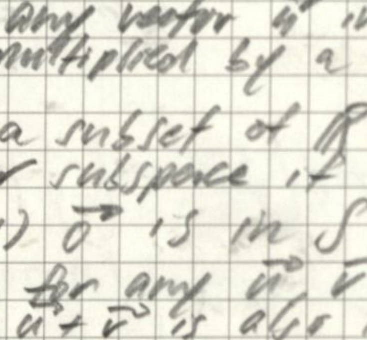
1) $\vec{0}$ is in S

2) for any $\vec{u}$, $\vec{v}$ in S, $\vec{u}+\vec{v}$ is also in S

3) for any $\vec{u}$ in S, $c\vec{u}$ is in S for any c

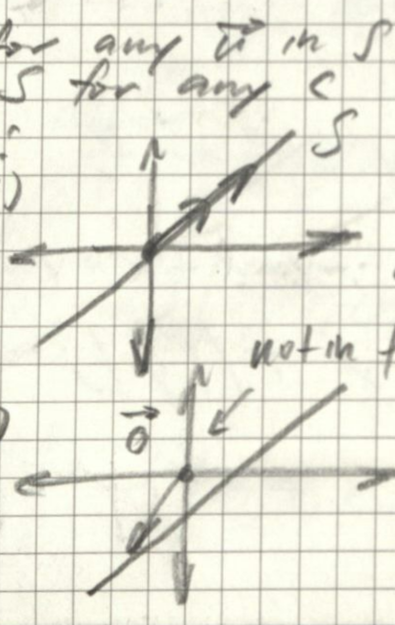
Ex: $\mathbb{R}^2$:

1)



line through origin is a linear subspace

2)



not in the line, scaling does not work

not a linear subspace

3)

not a linear subspace

Subspaces associated with any $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$$

Rows of A are vectors in $\mathbb{R}^n$
Columns of A are vectors in $\mathbb{R}^m$

Def. Row space of A is a span of row-vectors $A_1, \ldots A_m$
$= \{$linear combination of rows $A_1, \ldots A_m\}$

Def. Column space of A is a span of column-vectors $a_1, \ldots a_n$
$= \{$linear combination of columns $a_1, \ldots a_n\}$

In general, given any collection of vectors $\vec{v}_1, \ldots \vec{v}_k$ in $\mathbb{R}^n$ their span is a linear subspace

Notations: $row(A)$, $col(A)$

Consider a homogeneous linear system $A\vec{x} = \vec{0}$
Define $N = \{\vec{x} : A\vec{x} = \vec{0}\}$

1) $\vec{0} \in N$; $A\vec{0} = \vec{0}$

2) If $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0} \Rightarrow A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$
$\vec{x} \in N$ $\quad \vec{y} \in N \Rightarrow (\vec{x} + \vec{y}) \in N$

3) If $A\vec{x} = \vec{0}$ then $A(c\vec{x}) = cA(\vec{x}) = \vec{0}$
$\vec{x} \in N \Rightarrow c\vec{x} \in N$

$\Rightarrow$ N is a linear subspace of $\mathbb{R}^n$
N is called the null-space of A
denoted by $null(A)$

Sidenote:

$\in$ - belongs to
$\exists$ - exists
$\forall$ - any (every)

Recall - Linear independence/dependence of vectors

$\vec{v}_1, \dots \vec{v}_k$ are dependent if

$\exists c_1, \dots c_k$ not all zero

such that $c_1\vec{v}_1 + \dots c_k\vec{v}_k = 0$

If span$(\vec{v}_1, \dots, \vec{v}_k)$ and $\vec{v}_k$ is dependent w.r.t. $\vec{v}_1, \dots, \vec{v}_{k-1}$ $\Rightarrow$ $S = \text{span}(\vec{v}_1, \dots, \vec{v}_{k-1})$

Def. A basis of a linear subspace S is a collection of vectors if

1) it spans S
2) it is linearly independent

Ex 1) Recall:

in $\mathbb{R}^n$, we defined $\vec{e}_1, \vec{e}_2, \dots \vec{e}_n$

$$: \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

claim: $\vec{e}_1, \dots \vec{e}_n$ is a basis in $\mathbb{R}^n$

a) Any $\vec{x} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$

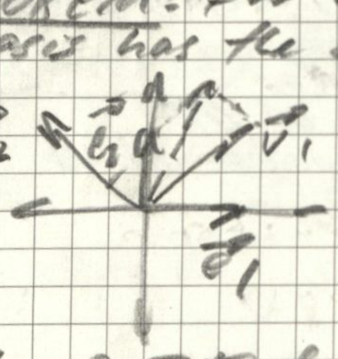
b) if $c_1\vec{e}_1 + c_2\vec{e}_2 + \dots c_n\vec{e}_n = \vec{0} = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix} = \vec{0}$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

$\Rightarrow \vec{e}_1, \dots \vec{e}_n$ independent.

Basis THEOREM: For a linear subspace $S \subseteq \mathbb{R}^n$, any basis has the same # of vectors in it

in $\mathbb{R}^2$:



Def. $\dim S$ = # of vectors in any basis of S

Ex. 20.

$$A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$$

OBSERVATION: Row space of A and of RREF of A are **THE SAME**

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & -2 & -6 & -7 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 1 & 4 & 4 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

1) 2 lin independent rows in R
they form a basis of $\text{row}(A) = \text{row}(R)$

2) if columns of A are dependent
$\Rightarrow x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{0}$

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

1) To find a basis of $\text{row}(A)$, we find the RREF of A and take <u>all</u> nonzero rows in it

2) To find a basis of $\text{col}(A)$, we pick columns of A corresponding to columns in RREF where leading 1s are

In our example, $\dim(\text{row}(A)) = 2$
$\dim(\text{col}(A)) = 2$

<u>Claim</u>. For any A, $\dim(\text{row}(A)) = \dim(\text{col}(A))$

<u>def.</u> $\dim(\text{row}(A)) = \dim(\text{col}(A)) \overset{\text{def}}{\Longrightarrow} \text{rank}(A)$.

$A\vec{x} = \vec{0} \iff R\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2x_2 - x_4 - \frac{1}{2}x_5 \\ x_2 \\ -3x_4 - \frac{7}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$\Rightarrow x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1/2 \\ 0 \\ -7/2 \\ 0 \\ 1 \end{bmatrix}$$

Lecture 7: 1.31.24

Linear Subspaces:

- span$(\vec{v}_1, \dots \vec{v}_k) = S$
- basis: if $\vec{v}_1, \dots \vec{v}_k$ are lin independent
  $\Rightarrow \dim S = k$
- A is $m \times n$
  $\Rightarrow$ row A = span of rows $\in \mathbb{R}^n$
  col A = span of columns $\in \mathbb{R}^m$
- $\text{Null}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$ in $\mathbb{R}^n$

Ex. A is a 3×5 matrix such that its RREF is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row(A) is made of all nonzero rows of R

$[1\ 0\ 1\ 0\ -1]$ & $[0\ 0\ 0\ 1\ 3]$

$\Rightarrow \dim \text{row}(A) = 2$

Basis for col(A) consists of columns # 1 and 4 of A

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ \& } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \dim \text{col} R = 2$$

To find Basis for Null(A) we solve the homogeneous $A\vec{x} = \vec{0} \Rightarrow R\vec{x} = \vec{0}$

$$R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0 \Rightarrow x_2, x_3, x_5 \text{ are free variables}$$

$$\left.\begin{array}{l} x_1 + x_3 - x_5 = 0 \\ x_4 + 3x_5 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -x_3 + x_5 \\ x_4 = -3x_5 \end{array}$$

General Solution:

$$\Rightarrow \begin{bmatrix} -x_3 + x_5 \\ x_2 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_5 \\ 0 \\ 0 \\ -3x_5 \\ x_4 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} x_5}_{\text{Basis for Null}(A)} \Rightarrow \dim \text{Null}(A) = 3$$

rank(A) = dim(A) = col(A) = 2

To find the basis for Null(A):
1) find RREF of A
2) write the general solution to $A\vec{x} = \vec{0}$ with free variables as parameters
3) Write it as a linear combination where free variables are coefficients
4) Vectors featured in 3 form the basis

dim Null(A) = 3
rank(A) = dim row(A) = dim col(A) = 2

Observation: 2 + 3 = 5 ⇒ # of columns

~~Cor~~ Thm: - For any $m \times n$ matrix A:
$\underbrace{\text{rank } A}_{\#\text{ of leading 1s in } R} + \underbrace{\text{dim Null}(A)}_{\text{nullity}} = n$

nullity ← # of free parameters

= # columns

Claim. For any linear equation $A\vec{x} = \vec{b}$, we have 3 options:
1) no solution
2) unique solution
3) ∞ solutions

Suppose $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$ and $\vec{x} \neq \vec{y}$
$A(\underbrace{\vec{x} - \vec{y}}_{\vec{x}_0}) = \vec{0}$

$\vec{x}_0 \in \text{Null}(A) \Rightarrow c\vec{x}_0 \in \text{Null}(A)$
$\Rightarrow A(\vec{x} + c\vec{x}_0) = A(\vec{x}) + cA(\vec{x}_0)$
$= \vec{b} + c \cdot \vec{0} = \vec{b}$

Ex. A is $n \times n$. Recall - A is invertable if there is A^{-1} s.t. $A^{-1}A = AA^{-1} = I$

Claim. A is invertable if any of the following is true
1) $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$
($A^{-1} \cdot A\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$)
2) RREF of A is I
3) rank(A) = n
4) Nullity of A is 0
5) columns of A form a basis in $\mathbb{R}^n$
6) columns of A are lin independent
7) " " about the rows of A
8)

Transposition of Matrices:

Def. If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & \ddots & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$

Def. $n \times n$ A is symmetric if $A^T = A$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$A\vec{x} = \vec{b}$ $\longrightarrow$ $[x_1 \ldots x_n] A^T = [b_1 \ldots b_m]$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

Property. $\text{rank}(A^T) = \text{rank}(A)$

Q. $\text{nullity}(A^T) \overset{?}{\neq} \text{nullity}(A)$

Let S be a linear subspace with a basis $\underbrace{\{\vec{v}_1, \ldots \vec{v}_k\}}_{B}$

$\Rightarrow$ Any $\vec{v}$ in S can be written

as $\vec{v} = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$

suppose $\vec{v} = d_1\vec{v}_1 + \ldots + d_k\vec{v}_k$

$\Rightarrow \vec{0} \; \underbrace{(c_1 - d_1)}_{0}\vec{v}_1 + \ldots + \underbrace{(c_k - d_k)}_{0}\vec{v}_k$

Notation.

$[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$

Ex. In $\mathbb{R}^3$,

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Lecture 8: 2.2.24

Note:

$$x_1\begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1}\end{bmatrix} + x_2\begin{bmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2}\end{bmatrix} + \dots + x_m\begin{bmatrix} v_{1m} \\ v_{2m} \\ \vdots \\ v_{nm}\end{bmatrix} = \underbrace{\begin{bmatrix} v_{11} & v_{12} & \dots & v_{1m} \\ v_{21} & v_{22} & & \\ \vdots & & \ddots & \\ v_{n1} & v_{n2} & & v_{nm}\end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m\end{bmatrix}}_{\vec{x}}$$

Basis:

$\mathbb{R}^n$, $B = \underbrace{\{\vec{v}_1, \dots \vec{v}_n\}}$

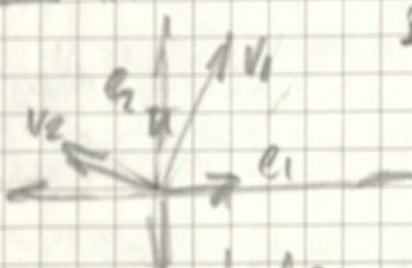
lin independent, span $\mathbb{R}^n$

Any $\vec{v}$ in $\mathbb{R}^n$ can be written as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

"defined uniquely"

$$\Rightarrow [\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n\end{bmatrix}$$

Ex: $\mathbb{R}^2$



Basis:

$$\mathcal{E} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$$

Pick $\vec{x} = \begin{bmatrix}3\\5\end{bmatrix}_{\mathcal{E}}$, want to find $[\vec{x}]_B = \begin{bmatrix}c_1\\c_2\end{bmatrix}$?

$$c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

$\rightsquigarrow$ Row Reduction $\rightsquigarrow \left[\begin{array}{cc|c}1 & -2 & 3\\2 & 1 & 5\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -2 & 3\\0 & 5 & -1\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 13/5\\0 & 1 & -1/5\end{array}\right]$

$$\Rightarrow [\vec{x}]_B = \begin{bmatrix}13/5\\-1/5\end{bmatrix}.$$

Similarly, we can find:

$$[\vec{e}_1]_B = \begin{bmatrix}1/5\\-2/5\end{bmatrix}, [\vec{e}_2]_B = \begin{bmatrix}2/5\\1/5\end{bmatrix}$$

Any $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \rightsquigarrow [\vec{x}]_B = [x_1\vec{e}_1 + x_2\vec{e}_2]_B$

$$= x_1[\vec{e}_1]_B + x_2[\vec{e}_2]_B = x_1\begin{bmatrix}1/5\\-2/5\end{bmatrix} + x_2\begin{bmatrix}2/5\\1/5\end{bmatrix}$$

$$= \begin{bmatrix}1/5 & 2/5\\-2/5 & 1/5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \Rightarrow [\vec{x}]_B = \begin{bmatrix}1/5 & 2/5\\-2/5 & 1/5\end{bmatrix}[\vec{x}]_{\mathcal{E}}$$

$\mathbb{R}^n$, $\mathcal{B} = \underbrace{\{\vec{u}_1, \dots \vec{u}_n\}}_{\text{lin independent}}$, $\mathcal{C} = \underbrace{\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}}_{\text{lin independent}}$ - basis

Any $\vec{x}$ in $\mathbb{R}^n$. Goal: Relate $[\vec{x}]_{\mathcal{B}}$ to $[\vec{x}]_{\mathcal{C}}$

Def. "A change-of-basis matrix" is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} \overset{\text{def}}{=} \left[[\vec{u}_1]_{\mathcal{C}} \; [\vec{u}_2]_{\mathcal{C}} \dots [\vec{u}_n]_{\mathcal{C}} \right]$$

Properties.

1) For any $\vec{x}$, $[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{B}}$

2) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is unique with this property

3) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$

Why? take $\vec{x} = x_1\vec{e}_1 + \dots x_n\vec{e}_n \iff [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Want $[\vec{x}]_{\mathcal{C}} = [x_1\vec{u}_1 + \dots + x_n\vec{u}_n] = x_1[\vec{u}_1]_{\mathcal{C}} + \dots + x_n[\vec{u}_n]_{\mathcal{C}}$

$$\Rightarrow \left[[\vec{u}_1]_{\mathcal{C}} \dots [\vec{u}_n]_{\mathcal{C}} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{B}} \quad 1), 2) \checkmark$$

$$[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{B}} \Rightarrow P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}$$

$P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$ $\quad P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$ $\quad$ 2) $\| \; P_{\mathcal{B}\leftarrow\mathcal{C}}$

4) $\mathcal{B}, \mathcal{C}, \mathcal{D}$ basis

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}} \cdot P_{\mathcal{C}\leftarrow\mathcal{B}}$$

5) $\mathcal{C} = \mathcal{E} = \{\vec{e}_1, \dots \vec{e}_n\}$, $\mathcal{B} = \left\{ \begin{bmatrix} u_{11} \\ \vdots \\ u_{n1} \end{bmatrix}, \dots \begin{bmatrix} u_{1n} \\ \vdots \\ u_{nn} \end{bmatrix} \right\}$

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} u_{11} & \dots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \dots & u_{nn} \end{bmatrix}$$

Recipe. Want to find $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} \cdot P_{\mathcal{E}\leftarrow\mathcal{B}}$

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = C^{-1} \cdot B \qquad = \underbrace{(P_{\mathcal{E}\leftarrow\mathcal{C}})}_{C}{}^{-1} \underbrace{P_{\mathcal{E}\leftarrow\mathcal{B}}}_{B}$$

B is a matrix whose columns are basis $\mathcal{B}$

C is a matrix whose columns are basis $\mathcal{C}$

$$P_{11}\vec{c}_1 + P_{21}\vec{c}_2 + \dots P_{n1}\vec{c}_n = \vec{b}_1 \quad \int \text{ aug matrix}$$

$$C\begin{bmatrix} P_{11} \\ P_{21} \\ \vdots \\ P_{n1} \end{bmatrix}_{\times 2} = \vec{b}_1{}_{\times 2} \Rightarrow \left[C \mid \vec{b}_1 \right]_{\times 2} \Rightarrow \left[C \mid \vec{b}_1\, \vec{b}_2 \dots \vec{b}_n \right]$$

We arrive at the aug matrix

$$[C \mid B] \xrightarrow[\text{reduction}]{\text{row}} [I \mid P] \leftarrow P_{\mathcal{C}\leftarrow\mathcal{B}}$$

Lecture 9. 2.5.24

$$\mathbb{R}^n = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\} \rightarrow \underline{\text{Vector Spaces}}: V, W$$

V is a vector space if we can define, for any $u, v \in V$ an operation of addition
$u, v \rightarrow u+v$ and an operation of multiplication by scalars $c, u \rightarrow cu$

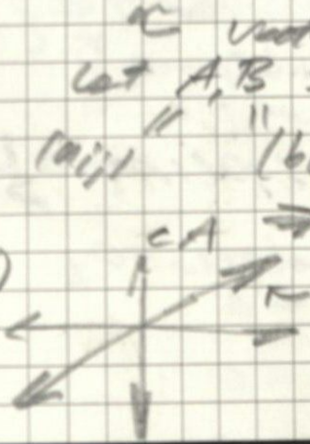
These two operations are subject to the following axioms:

1) $\forall u, v \in V,\ u+v \in V$
2) $u+v = v+u$ (communativity)
3) $(u+v)+w = u+(v+w)$ (associativity)
4) $\exists\ 0 \in V$ s.t. $0+u = u+0 = u$
5) $\forall u\ \exists (-u)$ s.t. $u+(-u) = 0$

6) $\forall u$ and $\forall c$, $cu \in V$
7) $(c+d)u = cu + du$ } distributivity laws
8) $c(u+v) = cu + cv$ }
9) $(cd)u = c(du)$
10) $1 \cdot u = u$

Ex. 1) $\mathbb{R}^n$

2) $M_{mn} = \forall\ m\times n$ matrices
$\in$ vector space
Let A, B be 2 $m\times n$ matrices
$A = (a_{ij})$, $B = (b_{ij})$ $\Rightarrow$ define $A+B = (a_{ij} + b_{ij})$
$cA \Rightarrow (ca_{ij})$

3)  $\leftarrow$ vector space

4)  not a vector space
$\forall u \nexists (-u)$

5) $\mathcal{P}_n \overset{def}{=} \{$all polynomials of degree $\leq n\}$

$= \{a_0 + a_1 x + a_2 x^2 + \dots a_n x^n = p(x)\}$

This is a vector space:

$$\begin{cases} p(x) = a_0 + a_1 x + \dots + a_n x^n \\ q(x) = b_0 + b_1 x + \dots + b_n x^n \end{cases}$$

$\Rightarrow p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$

$cp(x) = ca_0 + ca_1 x + \dots ca_n x^n$

6) $\{$all polynomials of degree exactly $n\}$

$= \{a_0 + \dots a_n x^n \,;\, a_n \neq 0\}$

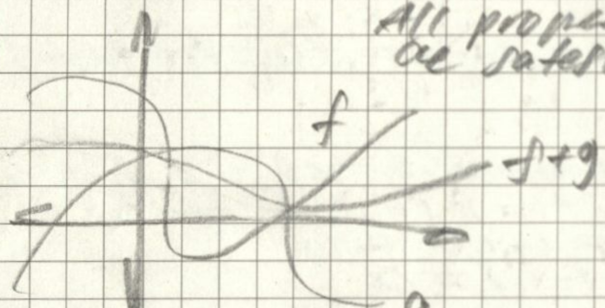
not a vector space $(\nexists\, 0 \in V)$

7) Vector space of all polynomials

8) All functions of the real line $- \mathcal{F} = \{f(x)\}$

$\mathcal{F}$ is a vector space: $f, g \Rightarrow$ define $(f+g)x = f(x) + g(x)$

All properties are satisfied $\Rightarrow (c \cdot f)x = c \cdot f(x)$



All notions we discussed for $\mathbb{R}^n$ can be generalized for any vector space V:

1) Subspace of a vector space: Def. W, subset of V, is a subspace if it is also a vector space w.r.t. the same $+$ and $\times$ by scalars

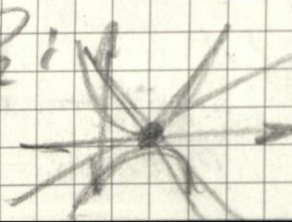
Claim. $W \in V$ is a vector subspace if $\forall\, u, v \in W$ and $\forall c$, $u + v \in W$ and $cu \in W$

Ex. 1) $M_{2,2} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}$. Consider $T = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right\}$ subspace

2) In $\mathcal{P}_n$ consider a subset

$W = \{p(x)$ of degree $\leq n \,;\, p(1) = 0\}$

for $\mathcal{P}_2$:



If $p(1) = 0$ and $q(1) = 0$

$\Rightarrow (p+q) = 0$

$\Rightarrow c \cdot p(1) = 0$

2) Linear span: If V is a vector space and $v_1, v_2, \dots v_n$ are vectors in V, then

$$\text{span}(v_1, v_2, \dots v_n) = \{c_1 v_1 + c_2 v_2 + \dots c_n v_n : c_1, c_2, \dots c_n \text{ are scalars}\}$$

$\leftarrow$ vector subspace of V

Ex. $\text{span}\{1, x, x^2\} = P_2$

$$\text{span}\{1, x, x^2\} = \{c_1 \cdot 1 + c_2 x + c_3 x^2\} = P_2$$

2) Is $P_2 = \text{span}\{1+x, x+x^2, 1+x^2\}$?

$\dim P_2 = 3$

Are $1+x, x+x^2, 1+x^2$ lin dependant?

If yes, then $\exists\, c_1, c_2, c_3$ s.t.

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$$

$$\underbrace{(c_1+c_3)}_{0} \cdot 1 + \underbrace{(c_1+c_2)}_{0} x + \underbrace{(c_2+c_3)}_{0} x^2 = 0$$

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned} \iff \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ RREF $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is independant

$\therefore$ yes

Lecture 10. 2.7.24

Vector Spaces: V, $v_1, \dots v_k \in V$

Def. span$(v_1, \dots v_k) = \{c_1 v_1 + \dots + c_k v_k : c_1, \dots c_k$ scalars$\}$

2) $v_1, \dots v_k$ are lin dependent if there are scalars $c_1, \dots c_k$ not all zero s.t. $c_1 v_1 + \dots c_k v_k = 0$

—— " —— independent if $c_1 v_1 + c_k v_k = 0$ implies $c_1 = 0, \dots c_k = 0$

3) Basis of V is $u_1, \dots u_n$ vectors $\in V$ s.t.

a) span$(u_1, \dots u_n) = V \Leftrightarrow v \in V$, $v = c_1 u_1 + \dots c_n u_n$ for some $c_1, \dots c_n$

b) $u_1, \dots u_n$ are lin independent

Ex. Recall $\mathcal{P}_n = \{a_0 + a_1 x + \dots a_n x^n : a_0, \dots a_n \in \mathbb{R}\}$

is set of all polynomials of degree $\leq n$

$\mathcal{B} = \{1, x, x^2, x^3, \dots x^n\}$

a) is clear

b) if $c_0 \cdot 1 + c_1 x + c_2 x^2 + \dots c_n x^n = 0 \in \mathcal{P}_n$

I.P. we can plug in $x = 0 \Rightarrow c_0 = 0$

Next, take a derivative:

$c_1 + 2c_2 x + 3c_3 x^2 + \dots n c_n x^{n-1} = 0$

For $x = 0 \Rightarrow c_1 = 0$

Next step, $2c_2 + 6c_3 x + \dots + n(n-1) c_n x^{n-1} = 0$

For $x = 0 \Rightarrow c_2 = 0$

⋮

After nth derivative:

$n! c_n \cdot 1 = 0 \Rightarrow c_n = 0$

$1, x, \dots x^n$ are lin independent

$\Rightarrow \mathcal{B}$ is basis

Corollary: $\mathcal{B} = \{1, (x-1), (x-1)^2, \dots (x-1)^n\}$ is also a basis

4) Every basis has the same number of vectors in it

Ex. $n = 2$, in $\mathcal{P}_2$, choose $\mathcal{B} = \{1, (x-1), (x-1)^2\}$

write $p(x) = x^2$ as a lin combination of $\mathcal{B}$

$x^2 = ((x-1)+1)^2 = (x-1)^2 + 2(x-1) + 1$

$= \underline{1} \cdot 1 + \underline{2} \cdot (x-1) + 1 \cdot (x-1)^2 \Longleftrightarrow [x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

For similar question in $\mathcal{P}_3$:

$$[x^3]_B = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \Leftrightarrow x^3 = ((x-1)+1)^3$$
$$\Rightarrow 1\cdot 1 + 3\cdot(x-1) + 3\cdot(x-1)^2 + 1\cdot(x-1)^3$$

b/c $(a+b)^2 = a^2+2ab+b^2$
and $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

__Generally:__ If $\mathcal{B} = \{u_1, \dots u_n\} \in V$,
then $\forall v \in V$ can be written as $v = c_1 u_1 + \dots c_n u_n$
and we write $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

__Properties:__

1) $[u+v]_B = [u]_B + [v]_B$
2) $[cu]_B = c[u]_B$
3) $[c_1 v_1 + \dots c_k v_k]_B = c_1 [v_1]_B + \dots c_k [v_k]_B$
4) $v_1, \dots v_k$ are lin independent in $V \Leftrightarrow [v_1]_B, \dots [v_k]_B$ are lin ind in $\mathbb{R}^n$
5) Any basis for V has the same number of vectors in it $\Rightarrow \dim V = \#$ of vectors in a basis

__Ex.__ $\dim \mathbb{R}^n = n$, $\dim \mathcal{P}_n = n+1$

$\dim \mathcal{P} = +\infty$
all polynomials

__Linear Transformations:__

V, W - vectors spaces
$T: V \to W$
$(\forall v \in V, T(v) = w \in W)$

__Def.__ T is a linear transformation from V to W if:

1) $T(v_1 + v_2) = T(v_1) + T(v_2)$
2) $T(cv) = cT(v)$

$\Rightarrow T(c_1 v_1 + \dots c_k v_k) = c_1 T(v_1) + \dots + c_k T(v_k)$

Ex. 1) Start with any $m \times n$ matrix A

let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $T = T_A$

$\underset{\text{in } \mathbb{R}^n}{T_A(v)} = \underbrace{A \cdot v}_{\text{in } \mathbb{R}^m}$

2) $V = \{$all continuous functions $f(t) \in [a, b]$ to $\mathbb{R}\}$

$W = \{\mathbb{R}\}$ $\quad T(f) \overset{\text{def}}{=} \int_a^b f(t)\,dt$

linear transformation

$T(f_1 + f_2) = \int_a^b (f_1(t) + f_2(t))\,dt$

$= \int_a^b f_1(t)\,dt + \int_a^b f_2(t)\,dt = T(f_1) + T(f_2)$

3) $V = \mathcal{P}_n$, $W = \mathcal{P}_{n-1}$, $T = \frac{d}{dx}$

$T(p(x)) = p'(x)$ ← linear transformation

4) $V = M_{mn}$, $W = M_{nm}$

$T(A) = A^T$

Lecture 11:   2.9.24

V, W vector spaces

$T: V \to W$ is a linear transformation if

$T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$

$\forall v_1, v_2 \in V$ and scalars c_1, c_2

Ex - $V = M_{2,3}$ = space of 2×3 matrices

$W = M_{3,2}$ = space of 3×2 matrices

$T(A \in V) = A^T \in W$

Properties:

1) $T(0) = 0_W$

$\vec{0} \in V \quad \vec{0} \in W$

2) $T(-u) = -T(u)$

3) $T(u-v) = T(u) - T(v)$

Superposition.

Recall. A - $m\times n$ matrix defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

If B - $k\times m$ matrix, we have linear trans.f. from $\mathbb{R}^m$ to $\mathbb{R}^k$

Then $B\cdot A$ is a $k\times n$ matrix that gives a lin. transf. from $\mathbb{R}^n$ to $\mathbb{R}^k$

In general,

$$V \xrightarrow{T} W \xrightarrow{S} U$$

$$v \longrightarrow T(v) \longrightarrow S(T(v))$$

Def. A superposition of S and T is

$(S\circ T)(v) \stackrel{\text{def}}{=} S(T(v))$

$S\circ T(c_1 v_1 + c_2 v_2) = S(T(c_1 v_1 + c_2 v_2))$

$= S(c_1 T(v_1) + c_2 T(v_2)) = c_1 S(T(v_1)) + c_2 S(T(v_2))$

$\therefore$ linear

Ex. $V = \mathbb{R}^2$, $W = \mathcal{P}_1$, $U = \mathcal{P}_2$

$\left.\begin{array}{l} T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x \\ S(p(x)) = xp(x) \end{array}\right\} \Rightarrow S\circ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$

$\Rightarrow S\left(T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) = S(a + (a+b)x)$

$= x(a + (a+b)x) = ax + (a+b)x^2$ still linear ✓

Invertible Linear Transformations:

1) Define, $\forall V$, the identity transformation $I_V(v) = v$. Eg. if $V=\mathbb{R}^n$, then $I_V = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

2) $T: V \to W$ is invertible if $\exists\ T': W \to V$ s.t. $T \circ T' = I_W$ and $T' \circ T = I_V$, then $T' = T^{-1}$

Ex. $T: \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A \cdot A' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

BUT. $A' \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $A' \neq A^{-1}$

Ex. $V = \mathbb{R}^2$, $W = P_1$

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x$

$T'(p_0 \cdot 1 + p_1 \cdot x) = \begin{bmatrix} p_0 \\ p_1 - p_0 \end{bmatrix}$

$(T \circ T')(p_0 + p_1 x) = T\left(\begin{bmatrix} p_0 \\ p_1 - p_0 \end{bmatrix}\right)$

$= p_0 + (p_0 + (p_1 - p_0))x = p_0 + p_1 x$

$\Rightarrow T \circ T' = I_{P_1}$

$(T' \circ T)\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T'(\underbrace{a}_{p_0} + \underbrace{(a+b)}_{p_1}x) = \begin{bmatrix} a \\ a+b-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow T' \circ T = I_{\mathbb{R}^2}$

Kernel and Range for Linear Transformations:

Nullspace Column Space

Fix $T: V \to W$ $\quad \vec{0} \in W$

Def. $\operatorname{Ker}(T) = \{ v \in V : T(v) = 0 \}$

$\operatorname{range}(T) = \{ w \in W : w = T(v)\ \forall v \in V \}$

Ex. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, $T(v) = Av$ where A is an $m \times n$ matrix

$\operatorname{Ker}(T) = \{ v \in \mathbb{R}^n : A \cdot v = \vec{0} \} = \operatorname{null}(A)$

$\operatorname{range}(T) = \{ w \in \mathbb{R}^m : Av = w\ \forall v \in \mathbb{R}^n \}$

$= \{ w \in \mathbb{R}^m : \underbrace{A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{linear combination of columns of } A} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \ \forall \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \}$

$= \{ w \in \mathbb{R}^m : w \text{ is linear comb. of columns of } A \}$

$= \operatorname{col}(A)$

Ex. $V = \mathcal{P}_3$, $W = \mathcal{P}_2$

$T = \frac{d}{dx} \iff T(p(x)) = p'(x)$

$\ker(T) = \{p(x) \in \mathcal{P}_3 : p'(x) = 0\} = \{c \cdot 1\}$

To find range(T), consider $p(x) = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3$ $(p(x) \in \mathcal{P}_3)$

then $T(p(x)) = p'(x) = a + bx + cx^2$

$\Rightarrow$ any polynomial in $\mathcal{P}_2$

$\ker(T) = \{c\}$, dim $= 1$

range$(T) = \mathcal{P}_2$, dim $= 3$

$1 + 3 = 4 = \dim \mathcal{P}_3$

Lecture 12. 2.12.24

V, W – vector spaces

$T: V \to W$ – linear transformation

$(T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2))$

Defined two linear subspaces:

$\ker(T) = \{v \in V : T(v) = 0\}$; subspace in V

range$(T) = \{w \in W \mid w = T(v)\ \forall v \in V\}$ subspace in W

For $m \times n$ matrix A we have a linear transformation:

from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$

then $\ker(T) = \text{null}(A)$, range$(T) = \text{col}(A)$

Ex. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix}$

$\ker(T_A) = \text{null}(A) = \left\{ \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} \forall c \right\}$

$= \left\{ c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim \text{null}(A) = 1$

OR:

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} c \\ 0 \\ c \end{bmatrix}$ – solution

$\Rightarrow$ nullity(A) + dim col(A) = 3

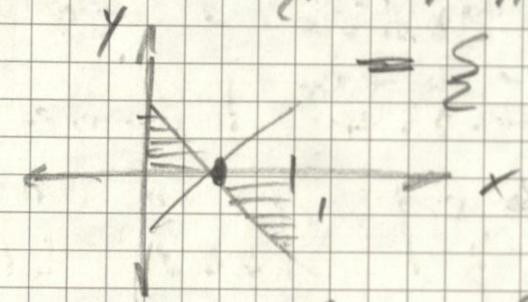
$\Rightarrow$ dim col(A) = 2 = dim $\mathbb{R}^2$

$\Rightarrow$ col(A) = range(T_A) = $\mathbb{R}^2$

Ex- $V = \mathcal{P}_1 = \{ p_0 + p_1 x \}$, $W = \mathbb{R}^1 = \mathbb{R}$, $S : V \to W$

$S(p(x)) \overset{def}{=} \int_0^1 p(x)\,dx$

$\ker(S) = \{ p(x) = p_0 + p_1 x : \int_0^1 (p_0 + p_1 x)\,dx = 0 \}$



$= \{ c(x - \frac{1}{2}) \}$

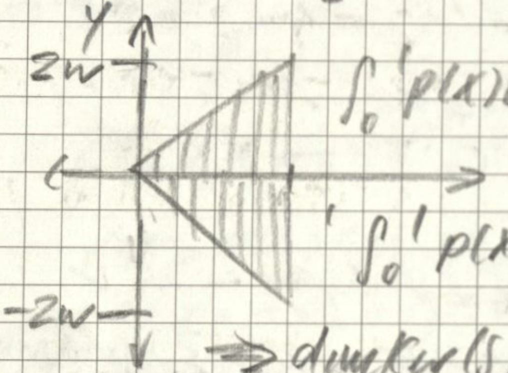
$\Rightarrow \dim \ker(S) = 1$

claim

for some?

$\text{range}(S) = \{ w = \int_0^1 (p_0 + p_1 x)\,dx : \forall\, p_0 + p_1 x \} = \mathbb{R}$

$\dim \text{range}(S) = 1$



$\int_0^1 p(x)\,dx = w$

$\int_0^1 p(x)\,dx = -w$

$\Rightarrow \dim \ker(S) + \dim \text{range}(S) = 2 = \dim \mathcal{P}_1$

Def. $T : V \to W$

nullity$(T) = \dim \ker(T)$

rank$(T) = \dim \text{range}(T)$

Rank Theorem: nullity(T) + rank(T) = dim source space?

compare: nullity(A) + rank$(A) = n$ (A is $m \times n$)

Ex- if A is $m \times n$ then A^T is $n \times m$

nullity(A) + rank$(A) = n$

nullity(A^T) + rank$(A^T) = m$

since $\dim \text{col}\, A^T = \dim \text{row}\, A$ vice versa:

rank(A) = rank(A^T)

$\Rightarrow$ nullity$(A^T) = m -$ rank(A^T)

$= m -$ rank$(A) = m - (n -$ nullity$(A))$

$= m - n +$ nullity(A)

Ex- $T: M_{2,2} \to M_{2,2}$ so 2×2 matricies are vectors

$T(A) \overset{def}{=} A\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ * vectors are defined by the space they live in *

$Ker(T) = \left\{ \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} : \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \right\}$

$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}a_{11}-a_{12} & -a_{11}+a_{12}\\ a_{21}-a_{22} & -a_{21}+a_{22}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow \begin{cases} a_{11} = a_{12} \\ a_{21} = a_{22}\end{cases} \Rightarrow A = \begin{bmatrix}a & a\\ b & b\end{bmatrix} = a\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$

$\Rightarrow \dim Ker(T) = nullity(T) = 2$

$rank(T) = \dim M_{2,2} - nullity(A) = 4 - 2 = 2$

$Range(T) = \left\{ \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} = \begin{bmatrix}a_{11}-a_{12} & -(a_{11}-a_{12})\\ a_{21}-a_{22} & -(a_{21}-a_{22})\end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix}\alpha_1 & -\alpha_1\\ \alpha_2 & -\alpha_2\end{bmatrix} \right\} = \left\{ \alpha_1\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} + \alpha_2\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix} \right\}$

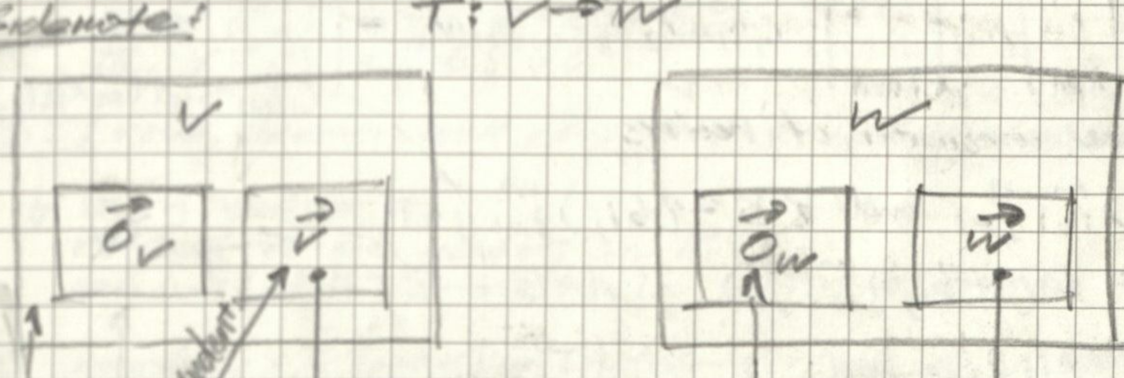
vector spaces → vectors are vectors
set of polynomials → polynomials are vectors
column space → columns of a matrix A are vectors
row space → rows of a matrix A are vectors

in above:

$B = T(A)$

$\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} = T\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix}a_{11}-a_{12} & -(a_{11}-a_{12})\\ a_{21}-a_{22} & -(a_{21}-a_{22})\end{bmatrix}$

Sidenote: $T: V \to W$



$Ker(T) = \{ \vec{v} \in V : T(\vec{v}) = \vec{0}_W \} \subseteq V$

$Range(T) = \{ \vec{w} \in W : \vec{w} = T(\vec{v}) \ \forall \vec{v} \in V \} \subseteq W$

dependent so not in range

Exam 1 Review!   2.12.24

2.1, 2.2: Gaussian elimination, row echelon form

Elementary Row Operations (EROs):
- I. replacement: $R_i \to R_i + cR_j$; $i \neq j$
- II. interchange: $R_i \leftrightarrow R_j$; $i \neq j$
- III. scaling: $R_i \to cR_i$; $c \neq 0$

Row Echelon Form (REF):
- I. All nonzero rows are above zero rows
- II. Each leading entry is left of all leading entries of lower rows

2.2: Gauss-Jordan elimination, free and leading variables

Reduced REF (RREF):
- I. Matrix is in REF
- II. All leading entries are 1
- III. Everything above/below leading entries is 0

Leading Variable: variable associated with the column corresponding to the first non-zero entry

Free Variable: variable associated with the column corresponding to columns without a non-zero entry of REF (II).

* columns that do not have a leading var have a free var*

2.3, 3.1, 3.3: spans, matrix operations

$\text{span}(\vec{v}_1, \dots \vec{v}_m) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_m\vec{v}_m : c_1, \dots, c_m \in \mathbb{R}\}$

spanning set: $\{\vec{v}_1, \dots \vec{v}_m\}$ s.t. $\text{span}(\vec{v}_1, \dots \vec{v}_m) = \mathbb{R}^n$

$\forall \vec{v}_1, \dots \vec{v}_m \in \mathbb{R}^n$ and $\forall c_1, \dots c_m \in \mathbb{R}^n$ where

$$\vec{v}_1, \dots, \vec{v}_m = \begin{bmatrix} x_{11} \\ \vdots \\ x_{m1} \end{bmatrix}, \dots, \begin{bmatrix} x_{1n} \\ \vdots \\ x_{mn} \end{bmatrix}, \; c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_m\vec{v}_m = \begin{bmatrix} c_1x_{11} + \dots + c_mx_{1n} \\ \vdots \\ c_1x_{m1} + \dots + c_mx_{mn} \end{bmatrix},$$

a.k.a. a linear combination of vectors.

$\forall A = (a_{ij})_{i=1}^{m}{}_{j=1}^{n}$ and $\forall B = (b_{ij})_{i=1}^{m}{}_{j=1}^{n}$

$$A + B = (a_{ij} + b_{ij})_{i=1}^{m}{}_{j=1}^{n} \; ; \; A + B = B + A$$

$$cA = (ca_{ij})_{i=1}^{m}{}_{j=1}^{n}$$

$$A \cdot B = a_{i1}b_{1j} + a_{i2} + b_{2j} + \dots a_{in} + b_{nj}$$

$$\Rightarrow \underset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \\ b_{n1} & & b_{np} \end{bmatrix}} = \underset{m \times p}{\begin{bmatrix} a_{11}b_{11} + \dots a_{1n}b_{n1} & \dots & a_{11}b_{1p} + \dots + a_{1n}b_{np} \\ \vdots & \ddots & \\ a_{m1}b_{11} + \dots a_{mn}b_{n1} & \dots & a_{m1}b_{1p} + \dots a_{mn}b_{np} \end{bmatrix}}$$

If $\underset{m\times n}{A}$, $\underset{n\times p}{B}$, $\underset{p\times q}{C}$:

$A\cdot B \neq B\cdot A$
$A\cdot(B\cdot C) = (A\cdot B)\cdot C$

If A is $n\times n$:

$A\cdot I = I\cdot A = A$

Also: if $A\vec{x} = \vec{b}$,

$A^{-1}\vec{b} = \vec{x}$

If $A\cdot B = B\cdot A = I$

$B = A^{-1}$ — A^{-1} only exists sometimes!
$A^{-1}\cdot A = I$

If A is 2×2:

$\rightarrow \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ — if 0, A^{-1} DNE! — if not I, A^{-1} DNE!

Also:

$[A \mid I] \xRightarrow{\text{EROs}} [I \mid A^{-1}]$

3.6: linear transformations

$T: \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \dashrightarrow T(\vec{x})$

$\forall\ \vec{u}, \vec{v}$ and $c \in \mathbb{R}^n$:
I. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
II. $T(c\vec{u}) = cT(\vec{u})$
$\Rightarrow T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \dots + c_mT(\vec{v}_m)$

Proof: $T(\vec{v}+\vec{v}') \overset{?}{=} T(\vec{v}) + T(\vec{v}')$
$T(c\vec{v}) \overset{?}{=} cT(\vec{v})$

Standard Matrix Representation: $T_A\ \forall A$
$\Rightarrow [T(\vec{e}_1)\ T(\vec{e}_2)\ \dots\ T(\vec{e}_n)]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}$

2.3, 3.5: linear independence, subspaces

$\vec{v}_1, \dots \vec{v}_k$ are dependent if $\exists\, c_1, \dots c_k$; $c \neq 0$
s.t. $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$. otherwise independent.

linear subspace: $S \subseteq \mathbb{R}^n$ s.t.
I. $\vec{0} \in S$
II. $\forall\ \vec{u}, \vec{v} \in S$, $\vec{u}+\vec{v} \in S$
III. $\forall\ \vec{u} \in S$, $c\vec{u} \in S\ \forall c$

3.5: row, column, null space of a matrix; basis for a subspace

$\forall$ $m\times n$ matrix A
rows of A are vectors in $\mathbb{R}^n$
columns of A are vectors in $\mathbb{R}^m$

row(A) = { linear combination of rows $A_1, \dots A_m$ }

col(A) = { linear combination of columns $a_1, \dots a_n$ }

row(A) = all non-zero rows of R (b/c of $\bar{R}R$)

col(A) = pivot columns of A

Null(A) = basis of homogeneous $A\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$

Basis: set of linearly independent vectors that span a subspace S

$col(A) = \{ \vec{w} \in \mathbb{R}^m : A\vec{v} = \vec{w} \; \forall \vec{v} \in \mathbb{R}^n \}$

$Null(A) = \{ \vec{v} \in \mathbb{R}^n : A\vec{v} = \vec{0} \in \mathbb{R}^m \}$

3.5: dimension, rank, nullity

dim: # of vectors in a basis for any linear space

$rank(A) = \dim col(A) = \dim row(A)$

$Nullity(A) = \dim Null(A)$

Rank theorem: $\underbrace{rank(A)}_{\text{\# of leading variables}} + \underbrace{Nullity(A)}_{\text{\# of free variables}} = \overbrace{n}^{\text{\# of columns}}$ - dim of source space

6.3: Coordinate systems in $\mathbb{R}^n$:

For linear space $\mathbb{R}^n$ with basis $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$
(where $\vec{v}_1, \dots \vec{v}_n$ are independent and span $\mathbb{R}^n$):

any $\vec{v}$ can be written as $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

$\Rightarrow [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, "coefficients of the basis vectors that create a vector $\vec{v}$"

Ex. in $\mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$

$\vec{e}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} \Rightarrow [\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix}1/2\\1/2\\-1/2\end{bmatrix}$

6.3: Change of Basis

$\mathbb{R}^n$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$, $\mathcal{C} = \{\vec{u}_1, \vec{u}_2, \dots \vec{u}_n\}$

$\mathcal{P}$ - matrix where $P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[\vec{v}_1]_{\mathcal{C}} \; [\vec{v}_2]_{\mathcal{C}} \cdots [\vec{v}_n]_{\mathcal{C}} \right]$

1) $[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{B}}$

2) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is a unique matrix

3) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertable and $P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$

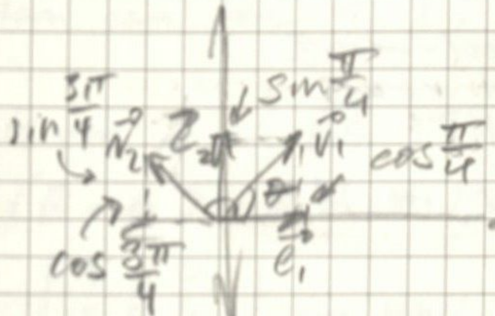
4) $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} \cdot P_{\mathcal{E}\leftarrow\mathcal{B}}$

where $P_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \\ v_{n1} & & v_{nn} \end{bmatrix}$

Also, $[\mathcal{C} \mid \mathcal{B}] \xrightarrow{EROs} [I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}]$

Lecture 13: Exam 1 Review 2.14.24

1) Write a matrix of a linear transformation in $\mathbb{R}^2$ which represents a counter-clockwise rotation by $\pi/4$.

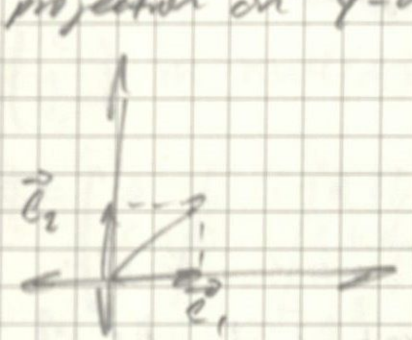


$$[T] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

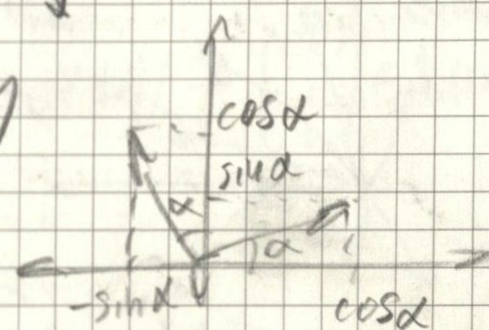
for arbitrary α

$$\rightarrow [T] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

2) projection on y-axis



$$[T] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Multiplying Matrices:

1) $A \cdot B$ only makes sense if # of columns of A = # of rows of B

2) A, B, C are square of the same size:

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

$A \cdot (B + C) = A \cdot B + B \cdot C$

$A \cdot B \neq B \cdot A \Rightarrow (A+B)^2 \neq A^2 + 2AB + B^2$

$\Rightarrow (A+B)(A+B) = A^2 + AB + BA + B^2$

Spaces:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix} \xrightarrow{RR} R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

1) row space of A is spanned by nonzero rows in the RREF

2) column space of A is spanned by columns of A that correspond to columns in in R that contain leading 1s

$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

$\text{row}(A) = \text{span}\left\{ [1 \ 0 \ -1], [0 \ 1 \ 0] \right\}$

For null(A) look at $\begin{bmatrix}1&0&-1&|&0\\0&1&0&|&0\end{bmatrix}$

solution to $A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$ is $\begin{bmatrix}x_3\\0\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\0\\1\end{bmatrix}$

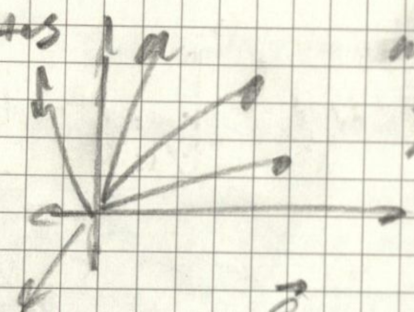
$\#$ no free variables, then null(A) = 0

Rank(A) = # of lin independent columns
= # of lin independent rows
= # of leading 1s in RREF

Rank(A) + Nullity(A) = # columns in A

3) $\left\{\begin{bmatrix}x\\y\end{bmatrix}, x\geq 0, y\geq 0\right\}$

violates



not a linear subspace

$\neq \vec{0}$, not ex.

4) $\mathcal{C}=\left\{\overset{\vec{c}_1}{\begin{bmatrix}1\\1\end{bmatrix}},\overset{\vec{c}_2}{\begin{bmatrix}1\\-1\end{bmatrix}}\right\}$, $P_{\mathcal{C}\leftarrow\mathcal{B}}=\left[[\vec{b}_1]_{\mathcal{C}}\ [\vec{b}_2]_{\mathcal{C}}\right]$

$\mathcal{B}=\{\vec{b}_1, \vec{b}_2\}$ $\quad\overset{\text{def}}{=}\begin{bmatrix}1&2\\2&1\end{bmatrix}$

$[\vec{b}_1]_{\mathcal{C}} \to \begin{bmatrix}1\\2\end{bmatrix} = 1\cdot\vec{c}_1 + 2\vec{c}_2$

$=\begin{bmatrix}3\\-1\end{bmatrix}$

$[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix}1\\-1\end{bmatrix} = 2\cdot\vec{c}_1 + 1\cdot\vec{c}_2 = \begin{bmatrix}3\\1\end{bmatrix}$

5) $\underset{A}{\begin{bmatrix}1&0&1\\0&1&1\\-1&0&1\end{bmatrix}}\left|\underset{I}{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}\right] \sim \begin{bmatrix}1&0&1\\0&1&1\\0&0&2\end{bmatrix}\left|\begin{matrix}1&0&0\\0&1&0\\1&0&1\end{matrix}\right] \sim \underset{I}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}\left|\underset{A^{-1}}{\begin{matrix}1/2&0&-1/2\\-1/2&1&-1/2\\1/2&0&1/2\end{matrix}}\right]$

* only square matrices are invertible

6) $\begin{bmatrix}1&2&k\\2&k^2&4\end{bmatrix}$ for which values of k do the columns not span $\mathbb{R}^2$

# of independent columns = # of independent rows

$\to \begin{bmatrix}1&2&k\\0&k^2-4&4-2k\end{bmatrix}$

$k^2-4=0$

$4-2k=0 \Rightarrow k=2$

inconsistent system → no solutions

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

→ consistent system

system can have 0, 1, or ∞ solutions
→ less leading 1s than rows in matrix

7) Given A and BA → find B

$$(B \cdot A) \cdot A^{-1} = B \cdot (A \cdot A^{-1}) = B \cdot I = B$$

8) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → null space contains everything

$\begin{bmatrix} c \\ c \\ c \end{bmatrix}$ in null(A) → $c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in null(A) → $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in$ null(A)

$$A = \begin{bmatrix} a & b & -(a+b) \\ c & d & -(c+d) \\ e & f & -(e+f) \end{bmatrix}$$

null$(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$
always a vector subspace

Lecture 14: 2.16.24

Exam 1: 99

Mean: 80 Our Section: 83.6

Median: 83 Our Section: 80

A − > 89.5

B − > 74.5

C − > 54.5

D − > 44.5

8. $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$

$[\vec{x}] = \begin{bmatrix}4\\6\\1\end{bmatrix} \Rightarrow \vec{x} = a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix}$

$a = 5$

$a + b = -2 \Rightarrow b = -7$

$a + b + c = 3 \Rightarrow c = 3 - a - b = 5$

12. $\mathcal{B} = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix} \right\}$

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}-6\\1\end{bmatrix}$

a) $-6\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}-6\\1\end{bmatrix} = \begin{bmatrix}-4\\-13\end{bmatrix}$

common mistake:

$\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix} \begin{bmatrix}-6\\1\end{bmatrix}$

b) $P_{\mathcal{C} \leftarrow \mathcal{B}}$: $[\mathcal{C} \mid \mathcal{B}] \Rightarrow \left[\begin{array}{cc|cc}1 & 3 & 1 & 2\\2 & 5 & 2 & -1\end{array}\right] \overset{\text{EROs}}{\Longrightarrow}$

$P_{\mathcal{C} \leftarrow \mathcal{B}} = \mathcal{C}^{-1}\mathcal{B} = \begin{bmatrix}1 & -13\\0 & 5\end{bmatrix}$

c) $[\vec{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1 & -13\\0 & 5\end{bmatrix}\begin{bmatrix}-6\\1\end{bmatrix} = \begin{bmatrix}-19\\5\end{bmatrix}$

5. A is 5×6 matrix, nullity$(A) = 4$

What is nullity(A^T)?

Rank Thm: $\text{nullity}(A^T) + \underset{\substack{\| \\ \text{rank}(A) \\ = 6-4 = 2}}{\text{rank}(A^T)} = \underbrace{\#\text{ columns of } A^T}_{5}$

$\Rightarrow \text{nullity}(A^T) = 5 - 2 = 3$

11. $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1+x_2\\x_2+x_3\\x_1\\x_3\end{bmatrix} = \begin{bmatrix}1&1&0\\0&1&1\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

a) $\mathbb{R}^3 \to \mathbb{R}^4$

b) $B = \begin{bmatrix}1&0&-1&0\\0&-1&0&1\end{bmatrix} \to S = B\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = [\ldots]$

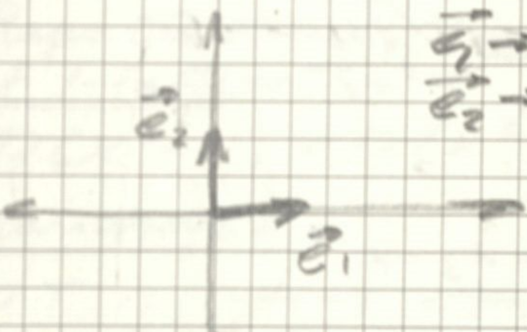
$\Rightarrow \begin{bmatrix}x_1 - x_3\\-x_2 + x_4\end{bmatrix}$

c) $(S \circ T)(\vec{x}) = S(T(\vec{x}))$.

$[S \circ T] = [S] \cdot [T] = \underset{2\times 4}{\begin{bmatrix}1&0&-1&0\\0&-1&0&1\end{bmatrix}}\underset{4\times 3}{\begin{bmatrix}1&1&0\\0&1&1\\1&0&0\\0&0&1\end{bmatrix}}$

$\Rightarrow \begin{bmatrix}0&1&0\\0&-1&0\end{bmatrix}$

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$



$\vec{e}_1 \to -\vec{e}_2$

$\vec{e}_2 \to \vec{e}_1$

$[T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$

7. $\vec{v}_1 = \begin{bmatrix}2\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix}-1\\-2\\1\end{bmatrix}$

$\{\vec{v}_1, \vec{v}_2\}$ are lin dep ✗

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are ~~lin dependent~~

$\vec{v}_3 = \vec{v}_2 - \vec{v}_1$ ✗

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ basis for $\mathbb{R}^3$ ✗

can only have n vectors in a basis for $\mathbb{R}^n$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$ ✗ same as b)

$\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ are lin ind ✓

Lecture 15: 4.19.24

$\underbrace{T: V \to W}_{\text{vector spaces}}$

$T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$

$\text{Ker}(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\} \subseteq V$

$\text{Range}(T) = \{\vec{w} \in W : \vec{w} = T(\vec{v})\} \subseteq W$

$\text{nullity}(T) = \dim \text{Ker}(T)$

$\text{rank}(T) = \dim \text{Range}(T)$

Rank Theorem: $\text{nullity}(T) + \text{rank}(T) = \dim V$

Ex. $V = \mathbb{R}^3$, $W = \mathcal{P}_2 = \{p_0 \cdot 1 + p_1 x + p_2 x^2\}$

$\mathcal{B} = \left\{ \underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$ $\mathcal{C} = \{1, x, x^2\}$

Inverse: T is invertible if $\exists T': W \to V$
where $T \circ T' = I_W$, $T' \circ T = I_V$

For Ex, $T: \mathbb{R}^3 \to \mathcal{P}_2$

$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = x_1 \cdot 1 + x_2 \cdot x + x_3 \cdot x^2$

$T^{-1}(p_0 + p_1 x + p_2 x^2) = \begin{bmatrix}p_0\\p_1\\p_2\end{bmatrix}$

$(T \circ T')(p_0 + p_1 x + p_2 x^2) = T\left(\begin{bmatrix}p_0\\p_1\\p_2\end{bmatrix}\right) = p_0 \cdot 1 + p_1 x + p_2 x^2$

$(T' \circ T)\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = T'(x_1 \cdot 1 + x_2 \cdot x + x_3 \cdot x^2) = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$\mathbb{R}^3$ and $\mathcal{P}_2$ are <u>isomorphic</u>

Def. T is called 1-to-1 if $\forall \vec{v}_1 \neq \vec{v}_2$,
$T(\vec{v}_1) \neq T(\vec{v}_2)$

T is called onto if $\text{Range}(T) = W$,
i.e. $\forall \vec{w} \in W \ \exists \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$

Ex. 1) $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -x_1 \\ 0 \end{bmatrix}$

Suppose $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$

$\Rightarrow T\left(\begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ Solve $T\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c_1 + c_2 \\ -c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to c_1 = 0,\ c_2 = 0$

$\Rightarrow$ 1-to-1

T is not onto since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ cannot be in Range(T)

Ex. 2) $D: \mathcal{P}_2 \to \mathcal{P}_1$

$D(p(x)) = p'(x)$

Onto: $D\left(p_0 x + \frac{p_1}{2} x^2\right) = p_0 + p_1 x$
arbitrary polynomial $\in \mathcal{P}_1$

Range$(D) = \mathcal{P}_1$

Not 1-to-1:

$D(x^2+1) = D(x^2)$

Observations:

1) T is 1-to-1 $\Leftrightarrow$ $\ker(T) = \{0\}$
why? If $\ker(T) \neq \{0\} \Rightarrow$ for some nonzero $\vec{v}$, $T(\vec{v}) = 0 = T(\vec{0})$
If T is not 1-to-1, we have
$T(\vec{v}_1) = T(\vec{v}_2) \Rightarrow T(\vec{v}_1 - \vec{v}_2) = 0 \Rightarrow \vec{v}_1 - \vec{v}_2 \in \ker(T)$

2) $T: V \to W$, $\dim V = \dim W = n$
T is 1-to-1 $\Leftrightarrow$ T is onto
why? If 1-to-1 $\Rightarrow$ nullity$(T) = 0$
Rank Thm: $0 + \underbrace{\text{Rank}(T)}_{n} = \dim V = n$
$\Rightarrow$ Range$(T) = W$

3) T is 1-to-1
$\Rightarrow T$ maps linearly ind sets into linearly ind sets
$\vec{v}_1, \ldots \vec{v}_k$ are lin ind
Let $c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \ldots c_k T(\vec{v}_k) = 0$
$\Rightarrow T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_k\vec{v}_k) = 0$
$\Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_k\vec{v}_k = 0$ since T is 1-to-1
$\Rightarrow c_1, c_2, \ldots c_k = 0$ since $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are lin ind.

4) If $\dim V = \dim W$ and T is 1-to-1 and $\vec{v}_1, \dots \vec{v}_n$ is a basis in V
$\Rightarrow T(\vec{v}_1), \dots, T(\vec{v}_n)$ is a basis in W

5) T is invertable $\Leftrightarrow T$ is both 1-to-1 and onto

6) T is invertable $\Rightarrow \dim V = \dim W$

__Def.__ Invertable transformation $T: V \to W$ is called an isomorphism and V and W are called isomorphic in this case

__Claim.__ If $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$

why? V has a basis $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\}$
we can write $\forall \vec{v} \in V$ as $\vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$

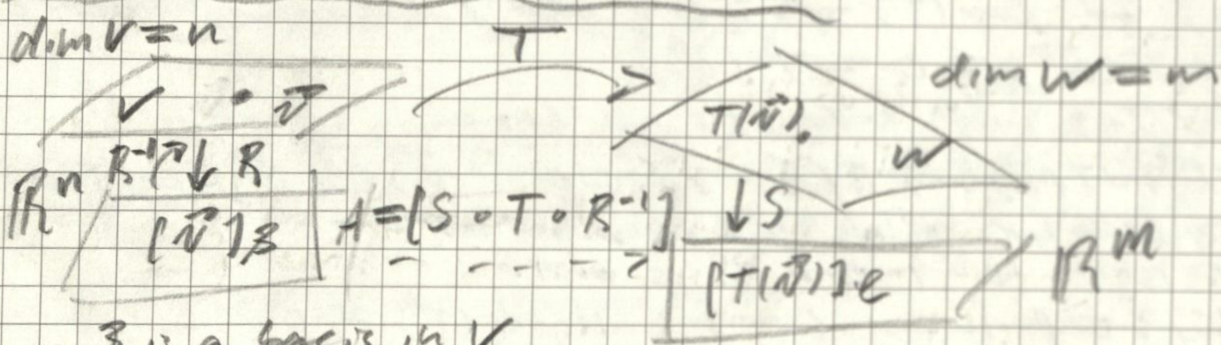
$$\Leftrightarrow [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

__Def.__ $T(\vec{v}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$

$T: V \to \mathbb{R}^n$ is an isomorphism
$T^{-1}: \mathbb{R}^n \to V$

$$T^{-1}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

__Matrix of a linear transformation:__



$\mathcal{B}$ is a basis in V
$\mathcal{C}$ is a basis in W

$$A = [S(T(R^{-1}))]$$

Notation for A is:

$* \; [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

__Example:__ $D: \mathcal{P}_3 \to \mathcal{P}_2$

$$\{1, x, x^2, x^3\} \qquad \{1, x, x^2\}$$

$$D \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ 2p_2 \\ 3p_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$D(p(x)) = p'(x)$
$D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2$

Lecture 16: 2.21.24

1. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{"}+\text{"} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1+1 \\ x_2+y_2+1 \end{bmatrix}$

1) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ ✓

2) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)+\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1+z_1+2 \\ x_1+y_1+z_1+2 \end{bmatrix}$ ✓

3) $\exists$ "$\vec{0}$" $= \begin{bmatrix} 0_1 \\ 0_2 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0_1 \\ 0_2 \end{bmatrix} = \begin{bmatrix} x_1+0_1+1 \\ x_2+0_2+1 \end{bmatrix} \Rightarrow \vec{0} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ ✓

4) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \exists \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$ s.t. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \vec{0} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$x_1+\hat{x}_1+1 = -1 \Rightarrow \hat{x}_1 = -x_1-2$ ✓
$x_2+\hat{x}_2+1 = -1 \qquad \hat{x}_2 = -x_2-2$

5) $c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} \in \mathbb{R}^2$ ✓

6) $c\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \stackrel{?}{=} \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}+\begin{bmatrix} cy_1 \\ cy_2 \end{bmatrix}$

$c\begin{bmatrix} x_1+y_1+1 \\ x_2+y_2+1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} cx_1+cy_1+1 \\ cx_2+cy_2+1 \end{bmatrix}$

not always true ✗ ...

only select boxes 7. and 8.

Matrix for a linear transformation:

$T: V \to W$

$\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\} \in V$

$\mathcal{C} = \{\vec{w}_1, \dots \vec{w}_n\} \in W$

We need to find a matrix A that sends a coordinate vector $[\vec{v}]_{\mathcal{B}}$ into a coordinate vector $[T(\vec{v})]_{\mathcal{C}}$

$[\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ b/c $\vec{v}_1 = 1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \dots + 0\cdot\vec{v}_n$ $\rightsquigarrow [T(\vec{v}_1)]_{\mathcal{C}}$

$[\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightsquigarrow [T(\vec{v}_2)]_{\mathcal{C}}$

$\vdots$

$[\vec{v}_n]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow [T(\vec{v}_n)]_{\mathcal{C}}$

Def.

$A = \Big[[T(\vec{v}_1)]_{\mathcal{C}} \ [T(\vec{v}_2)]_{\mathcal{C}} \cdots [T(\vec{v}_n)]_{\mathcal{C}}\Big]$

$= [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

Property:

$A \cdot [\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}$

Example: $T: \mathcal{P}_2 \to \mathbb{R}^2$

$\mathcal{B} = \{x^2, x, 1\}$ $\quad \mathcal{C} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$

$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[T(x^2)]_{\mathcal{C}} \; [T(x)]_{\mathcal{C}} \; [T(1)]_{\mathcal{C}} \right]$

$T(x^2) = \begin{bmatrix}0\\1\end{bmatrix} = c_1 \begin{bmatrix}1\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} = (-1)\begin{bmatrix}1\\0\end{bmatrix} + 1 \cdot \begin{bmatrix}1\\1\end{bmatrix}$

$[T(x^2)]_{\mathcal{C}} = \begin{bmatrix}-1\\1\end{bmatrix}, \; [T(x)]_{\mathcal{C}} = \begin{bmatrix}-1\\1\end{bmatrix}, \; [T(1)]_{\mathcal{C}} = \begin{bmatrix}0\\1\end{bmatrix}$

$\Rightarrow [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\forall \vec{v} \in \mathcal{P}_2 : \vec{v} = a + bx + cx^2$

$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \Rightarrow [T]_{\mathcal{C} \leftarrow \mathcal{B}} [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} -c-b \\ c+b+a \end{bmatrix}$

$[T(\vec{v})]_{\mathcal{C}} = [T(a+bx+cx^2)]_{\mathcal{C}} = \left[\begin{bmatrix} a \\ a+b+c \end{bmatrix} \right]_{\mathcal{C}}$

$= \left[(-b-c) \cdot \begin{bmatrix}1\\0\end{bmatrix} + (a+b+c) \begin{bmatrix}1\\1\end{bmatrix} \right]_{\mathcal{C}} \Rightarrow \begin{bmatrix} -b-c \\ a+b+c \end{bmatrix}$

$\therefore A \cdot [\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}$

Example: $T: \mathcal{P}_2 \to \mathcal{P}_2$ $\quad T(p(x)) = p(3x+1)$

$\mathcal{B} = \{1, x, x^2\}$ $\quad \{1, x, x^2\}$

$T(1) = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x) = 3x+1 = 1 + 3 \cdot x$ $\quad [T]_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 9 \end{bmatrix}$

$T(x^2) = (3x+1)^2 = 1 + 6x + 9x^2$

Example: $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ -a \\ b \end{bmatrix}$

$\mathcal{B} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\}$ $\quad \mathcal{C} = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\}$

Want: $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$

$T(\vec{v}_1) = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow [T(\vec{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$

$T(\vec{v}_2) = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow [T(\vec{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$

$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 8 \\ -3 & -4 \\ 2 & 1 \end{bmatrix}$

$T: U \to V, \; S: V \to W$

$\mathcal{B} \quad \mathcal{C} \quad \mathcal{C} \quad \mathcal{D}$ basis

$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} \cdot [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

Lecture 17: 2.23.24

$T: V \to W$

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, $\mathcal{C} = \{\vec{w}_1, \dots \vec{w}_m\}$

$\underbrace{[T]_{\mathcal{C} \leftarrow \mathcal{B}}}_{A} = [[T(\vec{v}_1)]_{\mathcal{C}} \dots [T(\vec{v}_n)]_{\mathcal{C}}] \Rightarrow A[\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}$

$T: \underset{\mathcal{B}}{U} \to \underset{\mathcal{C}}{V}$, $S: V \to \underset{\mathcal{D}}{W}$

$S \circ T: U \to W$

$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

Example: $T: \mathcal{P}_2 \to \mathcal{P}_2$, $T(p(x)) = p(3x+1)$

$S: \mathcal{P}_2 \to \mathbb{R}^2$, $S(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

$\Rightarrow S(T(p(x))) = S(p(3x+1)) = \begin{bmatrix} p(1) \\ p(4) \end{bmatrix}$

Basis:

$\mathcal{B} = \{1, x, x^2\} \in \mathcal{P}_2$

$\mathcal{C} = \{1, x, x^2\} \in \mathcal{P}_2$

$\mathcal{D} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \in \mathbb{R}^2$

$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

computed last lecture $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 16 \end{bmatrix}$

$p(x) = a + bx + cx^2$

$S(T(p(x))) = \begin{bmatrix} p(1) \\ p(4) \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+4b+16c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Recall: The Change of Basis Formula

$\mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}} [\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{C}} = [I \cdot V]_{\mathcal{C}}$

$\Rightarrow \mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}} = [I]_{\mathcal{C} \leftarrow \mathcal{B}}$

Important Case: $T: V \to V$, $\mathcal{B}, \mathcal{C} \in V$

suppose we know $[T]_{\mathcal{B} \leftarrow \mathcal{B}}$ and want $[T]_{\mathcal{C} \leftarrow \mathcal{C}}$

Note: $T = I \cdot T \cdot I$

$[T]_{\mathcal{C} \leftarrow \mathcal{C}} = \underbrace{[I]_{\mathcal{C} \leftarrow \mathcal{B}}}_{\mathcal{P}^{-1}_{\mathcal{B} \leftarrow \mathcal{C}}} [T]_{\mathcal{B} \leftarrow \mathcal{B}} \underbrace{[I]_{\mathcal{B} \leftarrow \mathcal{C}}}_{\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{C}}}$

Ex. $T:\mathbb{R}^2 \to \mathbb{R}^2$, $T: \begin{bmatrix}2&2\\3&1\end{bmatrix}$

$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$

$\mathcal{C} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\end{bmatrix} \right\}$ want $[T]_{\mathcal{C}\leftarrow\mathcal{C}}$.

$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}} \begin{bmatrix}2&2\\3&1\end{bmatrix} P_{\mathcal{B}\leftarrow\mathcal{C}}$

since $\mathcal{B} = \mathcal{E} \Rightarrow [\mathcal{B}|\mathcal{C}] = [I|\mathcal{C}] = [I|P_{\mathcal{B}\leftarrow\mathcal{C}}]$

$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = \frac{1}{5}\begin{bmatrix}3&2\\1&-1\end{bmatrix}\begin{bmatrix}2&2\\3&1\end{bmatrix}\begin{bmatrix}1&2\\1&-3\end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix}3&2\\1&-1\end{bmatrix}\begin{bmatrix}4&-2\\4&3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}20&0\\0&-5\end{bmatrix} = \begin{bmatrix}4&0\\0&-1\end{bmatrix}$

* makes matrix for T diagonal by picking the right basis

* picking the right basis is very hard

Def: $T: V \to V$ is diagonalizable if there is a basis $\mathcal{C} \subseteq V$ s.t. $[T]_{\mathcal{C}\leftarrow\mathcal{C}}$ is diagonal

Q. How to find $\mathcal{C}$

Requires Determinants!

Now $V = \mathbb{R}^n$, A is $n \times n$ matrix

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}$. Q. Is there a single expression in terms of a_{ij} that tells if A is invertable

1) $n=2$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow[\text{Reduction}]{\text{Row}} \frac{a_{21}}{a_{11}} R_1 - R_2$

want to get rid of a_{21} w/o effecting rest a_{ij} $= \begin{bmatrix} a_{11} & a_{12} \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} \end{bmatrix}$

A is invertable $\Leftrightarrow a_{11}a_{22} - a_{12}a_{21} \neq 0$

Def. $\det A \overset{\text{notation}}{=} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \overset{\text{def}}{=} a_{11}a_{22} - a_{12}a_{21}$

2) $n=3$

Same procedure of row reduction results in

$$\begin{bmatrix} a_{11} & * & * \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} & * \\ 0 & 0 & \frac{(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33})}{a_{11}a_{22} - a_{12}a_{21}} \end{bmatrix} \qquad \det A \overset{\text{def}}{=} ?$$

In Matrix Form! — — —

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{array}\right]$$

$+ \quad + \quad +$

* diagonalization stops working w/ $n > 3$

Sol!

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* allows $\det A$ of $\mathbb{R}^n$ to be expressed recursively with $\det A$ of $\mathbb{R}^{n-1}$

Lecture 18! 2.26.24

Ex.

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 2 & 2 & 2 \end{vmatrix} = 1\cdot 0\cdot 2 + 3\cdot 2\cdot 1 + 2\cdot 1\cdot(-2) - 1\cdot 0\cdot 2 - (-2)\cdot 2\cdot 1 - 2\cdot 1\cdot 3$$

$$\begin{matrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 2 & 2 & 2 \end{matrix}$$

$= 0$, matrix is non-invertable

A is 3×3,

$$|A| = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

Here A_{ij} = determinant of a submatrix of A obtained by crossing out row i and column j

Note!

If $\vec{u}, \vec{v}, \vec{w}$ are 3 vectors in $\mathbb{R}^3$

$$|\vec{u}\ \vec{v}\ \vec{w}| = (\vec{u}\times\vec{v})\cdot\vec{w}$$

By analogy: if $A = 4\times 4$ matrix, define

$$|A| = \det A = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - a_{14}A_{14}$$

In general, if we already defined 1×1, 2×2, ..., $(n-1)\times(n-1)$ determinants, then for an $n\times n$ matrix A, we get

$$|A| = \sum_{j=1}^{n} a_{1j}\underbrace{(-1)^{j+1}A_{1j}}_{=\pm 1} \qquad \text{Set } C_{ij} = (-1)^{i+j}A_{ij}$$

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij}$$

$$|A| = \sum_{\sigma} \operatorname{sign}(\sigma)\, a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n}$$

$\sigma = (\sigma_1, \dots, \sigma_n)$ permutation of $(1, 2, \dots, n)$

Ex. $\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 & 2 \\ -1 & 1 & 4 \\ 0 & 1 & -3 \end{vmatrix} - 0 + 3\begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 4 \\ 2 & 0 & -3 \end{vmatrix}$

$+1 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \ldots =$

Other ways to write the determinant:

$$|A| = \sum_{j=1}^{n} a_{\underline{i}j} C_{\underline{i}j} = \sum_{i=1}^{n} a_{i\underline{j}} C_{i\underline{j}}$$

$\forall i \in C$ $\qquad \forall j \in C$

(fix the rows or columns and cycle through other variables)

For Ex: $|A|$ also $=$

$(-1) \cdot C_{32} = (-1) \cdot (-1)^{3+2} \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix}$

* using the row or column with the most 0s, in this case column 2

* called the Laplace Expansion

Properties of Determinants:

1) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \sim \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$\rightarrow$ or columns

If you switch two rows in A, determinant gets multiplied by (-1)

2) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^T = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$|A^T| = |A|$

3) If you add to a row (or column) of A a multiple of another row (or column), the determinant does not change

4) If you multiply an entire row (or column) of A by the same constant c, the determinant also gets multiplied by c

For Ex.

$$\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & -2 & -2 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$-2\begin{vmatrix} 2 & 3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -2 \cdot 1 \cdot (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 \cdot (3-(-1)) = 8.$$

Ex. $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 1 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4-3 = 1$$

* for any square oriented pascal's triangle, the determinant is 1

Lecture 19: 2.28.24

Midterm 2, Thursday, March 7
will cover 6.1—6.6, 4.2

A is $n \times n$
Det(A) or $|A|$ is defined recursively by:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

where $C_{ij} = (-1)^{i+j} A_{ij}$, A_{ij} = det of $(n-1)\times(n-1)$ submatrix of A obtained by crossing out ith row and jth column of A

Properties:
1) Adding to a row/column of A a multiple of another row/column does not change $|A|$
2) Interchanging two rows/columns of A results in $|A|$ being multiplied by -1
3) Multiplying an entire row/column by a constant c $(c \neq 0)$ results in $|A|$ being multiplied by c

Each of the operations above cannot change nonzero det into zero det
These operations are done in row reductions

If A is invertible $\Rightarrow$ row reduces to $[I]$ where $|I| = 1$

$\Rightarrow A$ is invertible $\Leftrightarrow \det A \neq 0$.

Observations. Let A be a triangular matrix

i.e.
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \text{ or } A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 \cdots & 0 \\ * & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$$

$\det A = a_{11} \cdot a_{22} \cdot a_{nn}$

Main Property of Determinants:

A, B are $n \times n$ matrices

$|A \cdot B| = |A| \cdot |B|$, $|A+B| =$ who knows?

Linear System:

$A\vec{x} = \vec{b}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\vec{e}_1, \ldots \vec{e}_n$ are columns of (I)

Define:

$$I_i = \left[\vec{e}_1 \ldots \vec{e}_{i-1} \; \vec{x} \; \vec{e}_{i+1} \cdots \vec{e}_n \right]$$

$$= \begin{bmatrix} 1 & 0 & & 0 & x_1 & 0 & & 0 \\ 0 & 1 & & \vdots & x_2 & \vdots & & \vdots \\ \vdots & & \ddots & 1 & x_{i-1} & & & \\ 0 & 0 & & & x_i & 0 & & \\ & & & i-1 & \vdots & 1 & \ddots & \\ & & & & x_n & & & 1 \end{bmatrix} \Rightarrow |I|_i = x_i$$

$$A \cdot I_i = A \cdot \left(\vec{e}_1 \ldots \vec{e}_{i-1} \; \vec{x} \; \vec{e}_{i+1} \cdots \vec{e}_n \right), \; A\vec{e}_i = a_i$$

$$= \left[\vec{a}_1 \ldots \vec{a}_{i-1} \; \underline{\vec{b}} \; \vec{a}_{i+1} \ldots \vec{a}_n \right] = A(i)$$

$$|A \cdot I_i| = |A| \cdot |I_i| = |A| \cdot x_i$$

Conclusion:

$$x_i = \frac{|a_1 \ldots a_{i-1} \; b \; a_{i+1} \ldots a_n|}{|A|}$$

Cramer's Rule: solving $A\vec{x} = \vec{b}$,

to find x_i, we need to compute the determinant obtained from A by replacing ith column of A by $\vec{b}$ and then dividing by $|A|$.

Ex. $x+y-z=1$
$x+y+z=2$
$x-y=3$ $\iff \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{vmatrix} = 2\cdot(-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow 2\cdot(-1)\cdot(-2) = 4$$

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix}}{|A|} = \frac{1}{4}\left(0+3+2-(-3)-(-1)-0\right) = \frac{9}{4}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix}}{|A|} = -\frac{3}{4}, \quad x_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix}}{|A|} = -\frac{1}{2}$$

If A is invertable $\Rightarrow A\cdot A^{-1} = I$

$\Rightarrow |A|\cdot|A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$

$A^{-1} = [\vec{x}_1\ \vec{x}_2 \dots \vec{x}_j \dots \vec{x}_n]$, $\vec{x}_j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix}$

Want: $x_{ij} = ?$

$A\vec{x}_{ij} = \vec{e}_j \Rightarrow x_{ij} = \frac{1}{|A|} |\vec{a}_1 \dots \vec{a}_{i-1}\ \vec{e}_j\ \vec{a}_{i+1} \dots \vec{a}_n|$

ith column
jth row

$x_{ij} = \frac{1}{|A|}(-1)^{i+j} A_{ji} = \frac{(-1)^{i+j}A_{ji}}{|A|}$

← SWITCHED →

Ex. $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ & & \\ & & \end{bmatrix}$

Lecture 20: 3-1-24

A is 5×5 matrix, $|A|=7$

$|3\cdot A| = 3^5|A|$

Rule: if A is $n\times n$, then $|cA| = c^n|A|$

Recall: $T: V\to V$ and if $\mathcal{C}, \mathcal{B} \subset V$

$$[T]_{\mathcal{C}\leftarrow\mathcal{C}} = P^{-1}_{\mathcal{B}\leftarrow\mathcal{C}}[T]_{\mathcal{B}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}}$$

$V=\mathbb{R}^n$, $T(\vec{v}) = A\cdot\vec{v}$, A is $n\times n$

$\mathcal{B} = \mathcal{E} = \{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$

$\mathcal{C} = \{\vec{v}_1, \dots \vec{v}_n\}$ can be described as:

$C = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$

Call: $[T]_{\mathcal{C}\leftarrow\mathcal{C}} = \tilde{A}$

$\Rightarrow \tilde{A} = C^{-1}\cdot A\cdot C$

A and $\tilde{A}$ are similar

Def. A is diagonalizable if there is an invertible matrix C st. $C^{-1}AC$ is diagonal

Start with $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$A\vec{e}_1 = \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1\vec{e}_1$

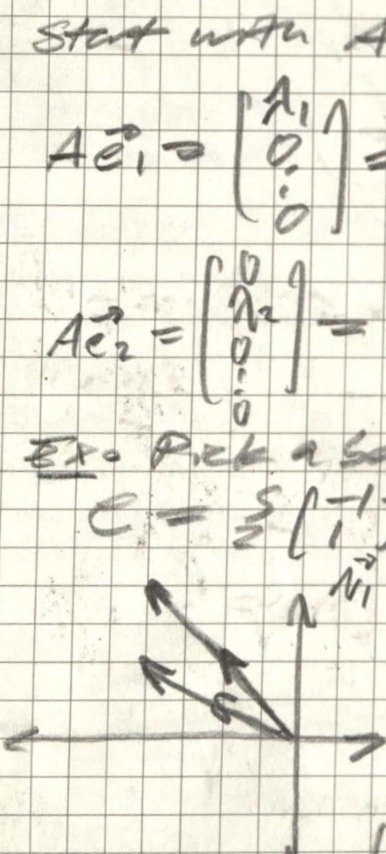
$\dots \ A\vec{e}_i = \lambda_i\vec{e}_i$

$A\vec{e}_2 = \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda\vec{e}_2$

Ex. Pick a basis in $\mathbb{R}^2$. Want to find matrix A

$\mathcal{C} = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ s.t. $A\vec{v}_1 = 2\vec{v}_1$, $A\vec{v}_2 = \frac{1}{2}\vec{v}_2$

$\vec{v}_1$ $\vec{v}_2$



$\Rightarrow [A]_{\mathcal{C}} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} = \tilde{A}$

$[A]_{\mathcal{C}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} A \cdot \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} = C^{-1}AC$

$\Rightarrow C[A]_{e \leftarrow e} C^{-1} = CC^{-1}ACC^{-1}$

$\Rightarrow A = C\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}C^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -3/2 & -7/2 \\ 3/2 & 9/2 \end{bmatrix}$

<u>Assume</u>, we know that for some basis $e = \{\vec{v}_1, \dots \vec{v}_n\}$, $[A]_{e \leftarrow e} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$C = [\vec{v}_1 \dots \vec{v}_n] \Rightarrow \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = C^{-1}AC$

$\Rightarrow C \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = A \cdot C$

$[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = A[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n]$

$A\vec{v}_1 = \lambda_1 \vec{v}_1$
$A\vec{v}_2 = \lambda_2 \vec{v}_2$
$\vdots$
$A\vec{v}_n = \lambda_n \vec{v}_n$

<u>Def:</u> if $\vec{v} \neq$ is s.t. $A\vec{v} = \lambda\vec{v}$ $\forall \lambda$ then λ is called an <u>eigenvalue</u> of A and $\vec{v}$ is called a corresponding <u>eigenvector</u>

How to find eigenvalues:

$A\vec{v} = \lambda\vec{v} \Longleftrightarrow A\vec{v} = \lambda \cdot I \cdot \vec{v}$

$\Longleftrightarrow \underbrace{(A - \lambda I)}\vec{v} = 0$

$\longrightarrow$ not invertable $\Rightarrow \det(A - \lambda I) = 0$

(nontrivial null space)

$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{22} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = \underbrace{\pm\lambda^n + \dots}_{p(\lambda):\ \text{polynomial of degree } n} \Rightarrow p(\lambda) = 0$ need to solve

Ex. $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$. To find eigenvalues, need to solve:

$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(1-\lambda) - (-1)\cdot 2$

$= (\lambda-4)(\lambda-1) + 2 = \lambda^2 + 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\Rightarrow$ solutions are $\lambda_1 = 2$, $\lambda_2 = 3$

For $\lambda_1 = 2$, look for $\vec{v}_1$ s.t. $\begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix} \vec{v}_1 = 0$

$\Rightarrow \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

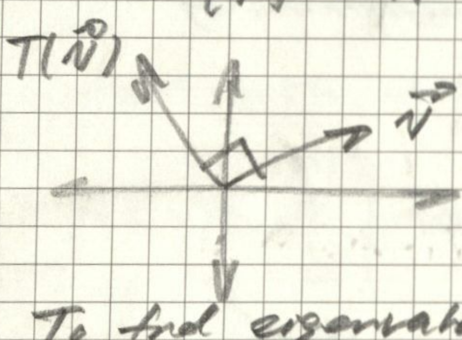
For $\lambda_2 = 3$, $\begin{bmatrix} 4-3 & -1 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The basis $C = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and:

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}}_{C^{-1}} \underbrace{\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{C} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Ex. In $\mathbb{R}^2$, T is a rotation counterclockwise by $90°$

$\Rightarrow [T] = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



To find eigenvalues to A we need to solve

$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$. no real solutions

$\therefore$ no real eigenvalues

Ex. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 0$

$A\vec{v} = 0 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

repeated eigenvalues

Lecture 21: 3.4.24

A — $n\times n$ matrix

λ — eigenvalue of A

$v\ (\neq 0)$ — corresponding eigenvector

if $Av = \lambda v$

$\Rightarrow \det(A-\lambda I) = |A-\lambda I| = 0$

$|A-\lambda I| = 0$ — characteristic equation of A

$p(\lambda)$ — a polynomial of degree n, called a characteristic polynomial

Possible Problems:

1) may not have real solutions to characteristic equations

2) may have repeated solutions (solutions with multiplicities)

Ex:

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ - Find eigenvalues and eigenvectors

1) char. eqn. $\rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(1-\lambda)\lambda + (1-\lambda) + (1-\lambda) = 0$

$\Rightarrow (1-\lambda)((1-\lambda)\lambda + 2) = 0$

$\Rightarrow (1-\lambda)(-\lambda^2+\lambda+2) = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(\lambda+1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -1$

2) Find eigenvectors

$\lambda_1 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\vec{v}_1 = 0 \xRightarrow{\text{EROs}} \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1 \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\vec{v}_3 = 0 \Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

3) Define $C = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$\Rightarrow C^{-1}AC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Ex. 2) $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(\lambda^2 - 6\lambda + 9)$

$\Rightarrow -(\lambda-3)^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 3$

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}\vec{v} = 0 \Rightarrow \vec{v} = \begin{bmatrix} a \\ b \\ -a \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$E_3 = \{\forall \vec{v} : (A - 3I)\vec{v} = 0\} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$

$\dim E_3 = 2$

$\lambda_1 = \lambda_2 = \lambda_3 = 3 \Rightarrow$ algebraic multiplicity of $\lambda = 3$, equal to 3

$E_3 \Rightarrow$ eigenspace

$\dim E_3 \Rightarrow$ geometric multiplicity

Problems:

3) For A to be diagonalizable, a geometric multiplicity must be equal to the algebraic multiplicity for every eigenvalue λ

Properties of Eigenvalues:

1) A is not invertible $\Leftrightarrow$ 0 is an eigenvalue $\Leftrightarrow \det A = 0 \Leftrightarrow \det(A - 0 \cdot I) = 0$

2) λ is an eigenvalue of A
$\Rightarrow \lambda^k$ is an eigenvalue of A^k $\forall k = 2, 3, \ldots$

$A^2 v = A(Av) \Rightarrow A(Av) - \lambda \Rightarrow Av = \lambda^2 Av$

$A^3 v = A(A^2 v) \Rightarrow A(A^2 v) = \lambda^2 Av \Rightarrow \lambda^3 v$

$\vdots$

3) A is invertible and $Av = \lambda v$
$\Rightarrow (A^{-1})v = \frac{1}{\lambda}v$. Why? $A(A^{-1}v) = v = A^{-1}(Av)$
$\Rightarrow \lambda A^{-1}v = v \Rightarrow A^{-1}v = \frac{1}{\lambda}v \quad = A^{-1}(Av) = \lambda A^{-1}v$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Find A^{13}

eigenvalues: $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow$ $\lambda^2 - \lambda - 2 = 0$
$(\lambda - 2)(\lambda + 1) = 0$
$\lambda_1 = -1, \lambda_2 = 2$

eigenvectors:

$\lambda_1 = -1 \rightsquigarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2 \rightsquigarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \vec{v}_2 = 0 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For $C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$, we have $C^{-1}AC = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

$C^{-1}A^2C = \underbrace{C^{-1}AC}_{\text{diag}} \cdot \underbrace{C^{-1}AC}_{\text{diag}} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 2^2 \end{bmatrix}$

$C^{-1}A^{13}C = \underbrace{C^{-1}AC \cdot C^{-1}AC \cdot C^{-1} \cdots C \cdot C^{-1}AC}_{13 \text{ times}}$

$\Rightarrow \begin{bmatrix} (-1)^{13} & 0 \\ 0 & 2^{13} \end{bmatrix}$

$\Rightarrow A^{13} = C \cdot \begin{bmatrix} -1 & 0 \\ 0 & 2^{13} \end{bmatrix} \cdot C^{-1}$

General Procedure:

A is $n \times n$ — want to diagonalize

1) Solve the characteristic equation $|A - \lambda I| = 0$
get solutions $\lambda_1, \lambda_2, \ldots \lambda_k$
of multiplicities $n_1, n_2, \ldots n_k$
s.t. $n_1 + n_2 + \ldots + n_k = n$

2). If $\forall \lambda_i$ is not real $\Rightarrow$ cannot diagonalize
otherwise $\forall \lambda$ compute basis of eigenspace
$E_{\lambda_i} = \{\vec{v} \mid A\vec{v} = \lambda_i \vec{v}\}$

3. If $\dim E_{\lambda_i} < n_i \Rightarrow$ cannot diagonalize

4. Otherwise, list ~~these bases~~

$C = [\underbrace{\vec{v}_{\lambda_1,1} \ldots \vec{v}_{\lambda_1,n_1}}_{n_1} \underbrace{\vec{v}_{\lambda_2,1} \ldots \vec{v}_{\lambda_2 n_2}}_{n_2} \ldots \underbrace{\vec{v}_{\lambda_k,1} \ldots \vec{v}_{\lambda_k,n_k}}_{n_k}]$

Then:

$$C^{-1}AC = \begin{bmatrix} \underbrace{\lambda_1 \ldots \lambda_1}_{n_1} & & & 0 \\ & \underbrace{\lambda_2 \ldots \lambda_2}_{n_2} & & \\ 0 & & \ddots & \\ & & & \underbrace{\lambda_k \ldots \lambda_k}_{n_k} \end{bmatrix}$$

Essentially:

$A\vec{v} = \lambda\vec{v} \rightarrow$ means transformation A is just a scalar transform of $\vec{v}$

$A\vec{v} - \lambda I\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0} \rightarrow$ want to find $\vec{v}$ s.t. $A = \lambda I$, meaning A is just scaling $\vec{v}$

$\det(\overbrace{A - \lambda I}^{\text{matrix}}) = 0$

$\rightarrow$ only way $A\vec{v} = \vec{0}$ $\forall \vec{v} \neq \vec{0}$ is if $\det(A) = 0$, meaning A squishes space into a lower dimension along $\vec{v}$

Problems:

1) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ results in $\lambda = -i, i$ which generally corresponds to rotation in the transformation

2) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ results in $(1-\lambda)^2 = 1$. There is only one eigenvalue but more than one eigenvector

Eigenbasis:

$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{bmatrix}$ matrix of this form represents a space where all of the basis vectors are eigenvectors with diagonal entries being their eigenvalues

$$\underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdots \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{100 \text{ times}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a^{100} & 0 \\ 0 & b^{100} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

you need to have enough eigenvectors to span the full space to have an eigenspace

Linear Algebra Exam 2 Review   3.5.24

6.1: Vector Spaces and Subspaces

Any space subject to these axioms:

1) $\forall \vec{u}, \vec{v} \in V,\ \vec{u}+\vec{v} \in V$
2) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutativity)
3) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (associativity)
4) $\exists\ \vec{0} \in V$ st. $\vec{0}+\vec{u} = \vec{u}+\vec{0} = \vec{u}$
5) $\forall\ \vec{u}\ \exists\ (-\vec{u})$ st. $\vec{u}+(-\vec{u}) = \vec{0}$

6) $\forall \vec{u} \in V$ and $\forall c,\ c\vec{u} \in V$
7) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
8) $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$ } (distributivity)
9) $(cd)\vec{u} = c(d\vec{u})$
10) $1 \cdot \vec{u} = \vec{u}$ ex. $\mathbb{R}^n, \mathcal{P}_n, \mathcal{M}_{mn}, \mathcal{F}_n$

Subspace:

$\forall W \subseteq V\ \ \forall \vec{u}, \vec{v} \in W$ and $\forall c,\ \vec{u}+\vec{v} \in W$ and $c\vec{u} \in W$

6.2: Linear independence, basis, dimension in a vector space

$\text{span}(\vec{v}_1, \dots \vec{v}_k) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k : c_1, \dots c_k \in \mathbb{R}\}$

$\vec{v}_1, \dots \vec{v}_k$ independent if $\exists\ c_1, \dots c_k = 0$ st.
$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_k\vec{v}_k = 0$

—— " —— dependent if $\nexists\ c_1, \dots c_k = 0$ st ——

$\forall \mathcal{B} = \{\vec{u}_1, \dots \vec{u}_n\} \in V,$

$\forall \vec{v} \in V \iff \vec{v} = c_1\vec{u}_1 + \dots c_n\vec{u}_n \iff [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Properties:

$[c_1\vec{v}_1 + \dots + c_k\vec{v}_k]_{\mathcal{B}} = c_1[\vec{v}_1]_{\mathcal{B}} + \dots + c_k[\vec{v}_k]_{\mathcal{B}}$

$\vec{v}_1 \dots \vec{v}_k$ independent $\in V \iff [\vec{v}_1]_{\mathcal{B}} \dots [\vec{v}_k]_{\mathcal{B}}$ independent $\in \mathbb{R}^n$

$\dim V \iff \#$ of $\vec{v} \in \mathcal{B} \in V$ ex. $\dim \mathcal{P}_n = n+1$

6.4: linear transformations

$T: V \to W \iff \forall \vec{v} \in V,\ T(\vec{v}) = \vec{w} \in W$ where:

$T(c_1\vec{v}_1 + \dots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \dots + c_kT(\vec{v}_k)$

Properties:

1) $T(\vec{0} \in V) = \vec{0} \in W$
2) $T(-\vec{u}) = -T(\vec{u})$
3) $T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v})$

Superposition:

$$V \xrightarrow{T} W \xrightarrow{S} U$$

$$\vec{v} \longrightarrow T(\vec{v}) \longrightarrow S(T(\vec{v})) = (S \circ T)(\vec{v})$$

Inverses: $\forall \vec{v},\ I_V(\vec{v}) = \vec{v}$

$T: V \to W$ is invertable if $\exists\ T': W \to V$
st. $T \circ T' = I_W$ and $T' \circ T = I_V \implies T' = T^{-1}$

6.2, 6.5: kernel and range, isomorphisms, coordinates in a vector space

$\mathrm{Ker}(T) = \{ \vec{v} \in V : T(\vec{v}) = \vec{0} \in W \} \subseteq V$

$\mathrm{Range}(T) = \{ \vec{w} \in W : \vec{w} = T(\vec{v}) \ \forall \vec{v} \in V \} \subseteq W$

$\mathrm{nullity}(T) = \dim \mathrm{Ker}(T)$

$\mathrm{rank}(T) = \dim \mathrm{Range}(T)$

$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim V$

T is 1-to-1 if $\forall \vec{v}_1 \neq \vec{v}_2 \Leftrightarrow T(\vec{v}_1) \neq T(\vec{v}_2)$

T is onto if $\mathrm{Range}(T) = W$

$\Leftrightarrow \forall \vec{w} \in W \ \exists \vec{v} \in V$ st. $T(\vec{v}) = \vec{w}$

Properties:

1) T is 1-to-1 $\Leftrightarrow \mathrm{Ker}(T) = \{0\}$

2) For $T: V \to W$ st. $\dim V = \dim W = n$
   T is 1-to-1 $\Leftrightarrow$ T is onto

3) T is 1-to-1 $\Leftrightarrow$ T maps independent sets to independent sets

4) T is 1-to-1 and $\dim V = \dim W$,
   $\mathcal{B} = \{\vec{v}_1, \dots \vec{v}_n\} \in V \Rightarrow \mathcal{C} = \{T(\vec{v}_1), \dots T(\vec{v}_n)\} \in W$

5) T is invertable $\Leftrightarrow$ T is 1-to-1 and onto

6) T is invertable $\Rightarrow \dim V = \dim W$

invertable $T: V \to W$ is an isomorphism and V and W are isomorphic

6.3, 6.6: change of basis in a vector space, matrix of a linear transformation. $T: V \to W$, $\mathcal{B} \in V$, $\mathcal{C} \in W$

$$A = [T]_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[T(\vec{v}_1)]_{\mathcal{C}} \ [T(\vec{v}_2)]_{\mathcal{C}} \ [T(\vec{v}_3)]_{\mathcal{C}} \right]$$

st. $A[\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}$

6.6: more on matrix of a linear transformation

$T: U \to V$, $S: V \to W$

$\mathcal{B} \in U$, $\mathcal{C} \in V$, $\mathcal{D} \in W$

$[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = [S]_{\mathcal{D} \leftarrow \mathcal{C}} \circ [T]_{\mathcal{C} \leftarrow \mathcal{B}}$

Also, $P_{\mathcal{C} \leftarrow \mathcal{B}} = [I]_{\mathcal{C} \leftarrow \mathcal{B}}$

So if: $T: V \to V$, $\mathcal{B}, \mathcal{C} \in V$

$T = I \cdot T \cdot I$

$\Rightarrow [T]_{\mathcal{C} \leftarrow \mathcal{C}} = [I]_{\mathcal{C} \leftarrow \mathcal{B}} [T]_{\mathcal{B} \leftarrow \mathcal{B}} [I]_{\mathcal{B} \leftarrow \mathcal{C}}$

$\Rightarrow [T]_{\mathcal{C} \leftarrow \mathcal{C}} = P^{-1}_{\mathcal{B} \leftarrow \mathcal{C}} [T]_{\mathcal{B} \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{C}}$

4.2: intro to determinants

$T: V \to V$ is diagonalizable if $\exists \mathcal{E} \in V$ st. $[T]_{\mathcal{E}\mathcal{E}}$ is diagonal

~~Laplace~~ Expansion:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

$\forall i \in C$ $\quad$ $\forall j \in C$

where $C_{ij} = (-1)^{i+j} A_{ij}$

st. A_{ij} is a subdeterminant obtained by crossing out row i and column j ($(n-1) \times (n-1)$)

Properties:

1) $R_i \to R_i + cR_j,\ i \neq j \Rightarrow |A| = |A|$

2) $R_i \leftrightarrow R_j,\ i \neq j \Rightarrow |A| = -|A|$

3) $R_i \to cR_i,\ c \neq 0 \Rightarrow |A| = c|A|$

If A is invertable $\xRightarrow{\text{EROs}} [I]$ st. $|I| = 1$

$\therefore$ A is invertable $\Leftrightarrow |A| \neq 0$

4) $|A^T| = |A|$, so all of above also applies to columns

4.2: more on determinants, Cramer's Rule

Let A be triangular:

$|A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$

$|A \cdot B| = |A| \cdot |B|$, $|A+B| \neq |A| + |B|$

$|cA| = c^n |A|$

For $A\vec{x} = \vec{b}$: $\quad \to$ triangular

$I_i = [\vec{e}_1 \ldots \vec{e}_{i-1}\ \vec{x}\ \vec{e}_{i+1} \ldots \vec{e}_n] \Rightarrow |I|_i = x_i$

$A \cdot I_i = [\vec{a}_1 \ldots \vec{a}_{i-1}\ \vec{b}\ \vec{a}_{i+1} \ldots \vec{a}_n]$

$\Rightarrow |A \cdot I_i| = |A| \cdot |I_i| = |A| \cdot x_i$

$$\Rightarrow x_i = \frac{|A \cdot I_i|}{|A|} = \frac{|a_1 \ldots a_{i-1}\ b_i\ a_{i+1} \ldots a_n|}{|A|}$$

Cramers Rule: solve $A\vec{x} = \vec{b}$ for x_i by dividing the determinant of the matrix obtained by replacing the ith column of A by $\vec{b}$ by $|A|$

A is invertable $\Rightarrow A \cdot A^{-1} = I \Rightarrow |A| \cdot |A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$ st. $A^{-1} = [\vec{x}_1\ \vec{x}_2 \ldots \vec{x}_j \ldots \vec{x}_n]$, $\vec{x}_j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{bmatrix}$

$A x_{ij} = \vec{e}_j \Rightarrow x_{ij} = \frac{1}{|A|} |\vec{a}_1 \ldots \vec{a}_{i-1}\ \vec{e}_j\ \vec{a}_{i+1} \ldots \vec{a}_n|$

$\Rightarrow x_{ij} = \frac{1}{|A|} (-1)^{i+j} A_{ji} = \frac{(-1)^{i+j} A_{ji}}{|A|}$ $\quad \hookrightarrow$ jth row, ith column

Lecture Notes: Exam 2 Review 5.6.24

1) $\mathcal{P}_3 = \{p(x) = a + bx + cx^2 + dx^3\}$

$H = \{p(x) \in \mathcal{P}_3 \mid p(-1) = 0\}$

$p(-1) = 0 \Longleftrightarrow p(x) = (x+1)(a + bx + cx^2)$

$\dim \mathcal{P}_3 = 4$, $\dim H = 3$

Notion: V vector space, $\dim V = n$. A basis of V contains n lin independent vectors

$\Longleftrightarrow \vec{v}_1, \ldots, \vec{v}_n$ – basis if $\text{span}(\vec{v}_1, \ldots, \vec{v}_n) = V$

I.P. if $\dim V = 5$, then 3 vectors in V can be independent but cannot span V

and 7 vectors can span V but cannot be independent

H is a subspace for V if for any $\vec{v}$ in H:
- $c\vec{v}$ is in H
- $\vec{u} + \vec{v}$ is in H

Always subspaces:

A is a matrix → null(A) is a subspace

T is a transformation → Ker(T) and Range(T)

in V   in W

Set of solutions of $A\vec{x} = \vec{b}$ is a subspace only if $\vec{b} = \vec{0}$

1) $p(x) = \underbrace{a(x+1)} + b\underbrace{(x+1)x} + c\underbrace{(x+1)x^2}$

basis vectors

4) I. rank(T) + nullity(T) = dim V = 5 for $T: V \to W$

II. true by definition   false

III. false, range is always a subspace of target space

T is 1-to-1 $\Longrightarrow$ Ker(T) = $\{0\}$, nullity(T) = 0

T is onto if Range(T) = W $\Longleftrightarrow \forall \vec{w} \in W \; \exists \vec{v} \in V$ s.t. $T(v) = w$

Ex- $T: M_{2\times2} \to \mathbb{R}$

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$ onto

If T is onto $\Longrightarrow \dim W \leq \dim V$

8) for linear T

$T(a+b) = T(a) + T(b)$

$T(ca) = cT(a)$

9) V, $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_n\}$

a. $P_{\mathcal{B}\leftarrow\mathcal{C}} = [[\vec{w}_1]_{\mathcal{B}} \dots [\vec{w}_n]_{\mathcal{B}}]$

$\vec{w}_1 = 1 - x + x^2 = (1-x) + x^2 = \vec{v}_1 + \vec{v}_3 \Rightarrow [\vec{w}_1]_{\mathcal{B}} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

$\vec{w}_2 = 1 + 3x = c_1(1-x) + c_2(x-x^2) + c_3x^2$

$\Rightarrow [\vec{w}_2]_{\mathcal{B}} = \begin{bmatrix}1\\4\\4\end{bmatrix}$ $\Rightarrow c_1 = 1 \Rightarrow -c_1 + c_2 = 3 \Rightarrow c_2 = 4$

$\Rightarrow -c_2 + c_3 = 0 \Rightarrow c_3 = 4$

$\vec{w}_3 = 2 - x - 2x^2 = c_1(1-x) + c_2(x-x^2) + c_3x^2$

$= c_1 + (c_2 - c_1)x + (c_3 - c_2)x^2$

$\Rightarrow c_3 = c_2 = -2 \Rightarrow c_1 \dots$

$[\vec{w}_3]_{\mathcal{B}} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 1 & 2\\0 & 4 & 1\\1 & 4 & -1\end{bmatrix}$

b. $[p(x)]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[p(x)]_{\mathcal{C}} = \begin{bmatrix}1 & 1 & 2\\0 & 4 & 1\\1 & 4 & -1\end{bmatrix}\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\3\\7\end{bmatrix}$

$p(x) = (1-x) + 3(x-x^2) + 7x^2$

9) a. $|A| = 2\begin{vmatrix}1 & 1 & 1 & 0\\-3 & 0 & 0 & \sqrt{91}\\1 & 1 & 0 & 11000\\1 & 1 & 1 & 1\end{vmatrix} = 2\begin{vmatrix}1 & 1 & 1 & 0\\-3 & 0 & 0 & \sqrt{91}\\1 & 1 & 0 & 11000\\0 & 0 & 0 & 1\end{vmatrix}$

$= \dots = 2\begin{vmatrix}-3 & 0\\1 & 1\end{vmatrix} = -6$

b. $(A^{-1})_{55} = (-1)^{5+5}\dfrac{A_{55}}{|A|} = \dfrac{-3}{-6} = \dfrac{1}{2}$

<u>Notion</u>. $(A^{-1})_{3,4} = (-1)^{3+4}\dfrac{A_{4,3}}{|A|}$

A_{ij} = sub<u>determinant</u> by definition

4) $T(f(t)) = f(2)$

$\ker(T) = \{$all poly $f(t)$ of degree ≤ 3 | $f(2) = 0\}$

$= \{(t-2)(a + bt + ct^2)\}$

Lecture Notes: Exam 2 3.8.24

A - > 89.5 Mean-Median: 79

B - > 74.5

C - > 59.5

D - > 49.5

Q. $T: P_1 \to P_1$, $T(1+2x) = 1-2x$, $T(2+3x) = 2+x$

$T(3+2x)$?

$(3+2x) = c_1(1+2x) + c_2(2+3x)$

$3+2x = c_1 + 2c_2 + (2c_1 + 3c_2)x$

$c_1 + 2c_2 = 3$ $\Rightarrow$ $c_1 + 2c_2 = 3$ $\Rightarrow$ $c_2 = 4$

$2c_1 + 3c_2 = 2$ $\quad 0 - c_2 = -4$ $\quad c_1 = -5$

Q. $S: M_{22} \to M_{22}$ rank(S) + nullity(S) = 4

$S(A) = A - A^T$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$S(A) = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$

$\text{Ker}(S) = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \right\}$

$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Q. you can have a span of any set of vectors

Q. Ker(T) is always a vector space; also Null(A)

Range(T) is always a vector space

solution $A\vec{x} = \vec{b}$ not always a vector space

— only when $A\vec{x} = \vec{0}$

Q. $\mathcal{B} = \{x^2+3, x-4, 1\}$, $\mathcal{C} = \{1-x^2, x^2+x, x^2\}$

$P_{\mathcal{C} \leftarrow \mathcal{B}} = \left[[x^2+3]_{\mathcal{C}} \; [x-4]_{\mathcal{C}} \; [1]_{\mathcal{C}} \right]$

Lecture Notes: 3.18.24

Similarity of Matrices:

- $n\times n$ matrices A and B are similar if, for some invertable P, $B = P^{-1}AP$
- if $B = D$ is a diagonal matrix, then A is diagonalizable

To find if A is diagonalizable and to implement diagonalization:

1) Write the characteristic eq-n for A:
$\det(A - \lambda I) = 0 \iff \lambda^n + p_n\lambda^{n-1} + \ldots + p_0 = 0$
If the equation has non-real solutions
$\Rightarrow$ not diagonalizable

2) If all solutions are real $\iff$ rewrite characteristic equation as
$(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\ldots(\lambda-\lambda_k)^{n_k} = 0$
with all $\lambda_1, \ldots \lambda_k \in \mathbb{R}$
λ_1 has algebraic multiplicity n_1
$\vdots$
λ_k —— " —— n_k

ex. $\lambda^3(\lambda-1)^2(\lambda+1) = 0$
solutions: 0 alg. mult. 3
1 2
-1 1

3) For every eigenvalue λ_i, set up an equation to find eigenvectors
$A\vec{v} = \lambda_i\vec{v}$ $(\vec{v} \neq 0)$
$\Rightarrow \begin{bmatrix} a_{11}-\lambda_i & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & a_{nn}-\lambda_i \end{bmatrix}\vec{v} = 0$ $\rightarrow$ forms the null space

4) Find all solutions $\Rightarrow$ form a subspace
$E_{\lambda_i} = \{\vec{v} \mid A\vec{v} = \lambda_i\vec{v}\}$
called eigenspace for eigenvalue λ_i
$\dim E_{\lambda_i}$ is a geometric multiplicity of λ_i

If $\dim E_{\lambda_i} < n_i \Rightarrow A$ is not diagonalizable
$\hookrightarrow$ algebraic multiplicity

4) If dim $E_{\lambda_i} = n_i$, pick any basis for E_{λ_i}
$\{\vec{v}_{\lambda_i,1}, \vec{v}_{\lambda_i,2}, \dots, \vec{v}_{\lambda_i,n_i}\}$
and build a matrix

$$P = \left[\vec{v}_{\lambda_1,1} \dots \vec{v}_{\lambda_1,n_1}\ \vec{v}_{\lambda_2,1} \dots \vec{v}_{\lambda_2,n_2} \dots \vec{v}_{\lambda_k,1} \dots \vec{v}_{\lambda_k,n_k}\right]$$

5)

$$P^{-1}AP = \begin{bmatrix} \underbrace{\lambda_1 \ddots \lambda_1}_{n_1} & & & \\ & \underbrace{\lambda_2 \ddots \lambda_2}_{n_2} & & \\ & & \ddots & \\ & & & \underbrace{\lambda_k \ddots \lambda_k}_{n_k} \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

characteristic Eqn: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda - 1)^3 = 0$, $\lambda_1 = 1$, $n_1 = 3$

Find $E_1 = \{A\vec{v} = \vec{v}\} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

$E_1 = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

$\dim E_1 = 2 < n_1 = 3$

not diagonalizable b/c P will not be $n \times n$

<u>Properties of Similarity</u> $(A \sim B)$

1) $A \sim B$ and $B \sim C \Rightarrow A \sim C$

$C = P^{-1}BP = P^{-1}(Q^{-1}AQ)P$

$\Rightarrow (P^{-1}Q^{-1})A(QP) = (QP)^{-1}A(QP)$

$((P^{-1}Q^{-1})QP = I)$

2) If $A \sim B$ and A is diagonalizable, then B is also diagonalizable

3) If $A \sim B \Rightarrow \det(A - \lambda I) = \det(B - \lambda I)$

$\det(B - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}P)$

$= \det(P^{-1}(A - \lambda I)P) = \det(P^{-1}) \det(A - \lambda I) \det P$

In Particular, if $A \sim B$

$\Rightarrow$ A and B have the same eigenvalues with the same multiplicities

$A \sim B \not\Leftarrow \det(A-\lambda I) = \det(B-\lambda I)$

* only works in one direction

Ex- $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has same characteristic eqn

as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ but $A \not\sim I$

4) If $\lambda_1, \dots \lambda_n$ are eigenvalues of A then $\det A = \lambda_1 \lambda_2 \cdot \dots \cdot \lambda_n$ b/c $A \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$\det A = \det \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

5) $A \sim B \Leftrightarrow B = P^{-1}AP$

$\Rightarrow B^k = P^{-1}A^kP$ for $k = 1, 2, \dots$

If A is invertible $\Rightarrow B^k = P^{-1}A^kP$ for all k

<u>Non-Diagonalizable Matrices</u> : $n = 2$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, what is the simplest matrix A is similar to?

1) A has two distinct and real eigenvalues λ_1, λ_2

$\Rightarrow A \sim \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

2) A has a real repeated eigenvalue λ_1

$\Rightarrow \det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{22} & a_{22}-\lambda \end{bmatrix} = (\lambda - \lambda_1)^2$

algebraic multiplicity = 2

If: geometric multiplicity = 2 $\Rightarrow A \sim \begin{bmatrix} \lambda_1 & \\ & \lambda_1 \end{bmatrix} = \lambda_1 I$

$\Rightarrow A = \lambda_1 I$

If: geometric multiplicity = 1

$\Rightarrow A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

3) A has complex eigenvalues:

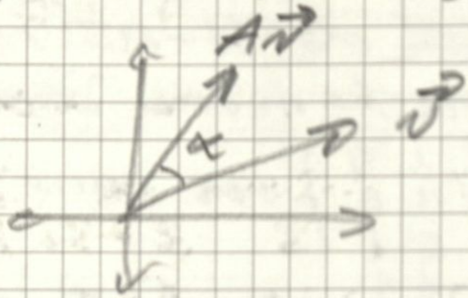
Ex- $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Lecture Notes: 3-20-24

$n=2$, what to do if there are no real eigenvalues?
Eigenvector v with eigenvalue λ satisfies $Av=\lambda v$

Recall:

$$A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$



Ex:

$$A=\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$$

char. Eqn $= \begin{vmatrix}a-\lambda & -b\\ b & a-\lambda\end{vmatrix}=0 \Leftrightarrow (\lambda-a)^2+b^2=0$

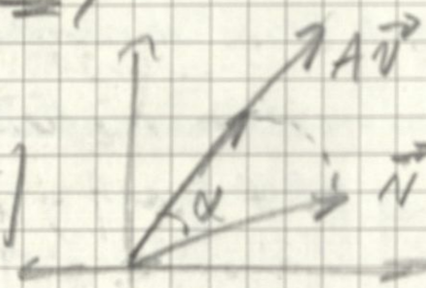
$\Leftrightarrow (\lambda-a)^2=-b^2 \Leftrightarrow \lambda-a=\pm bi$

$\Rightarrow \lambda=a\pm bi$

Define $r=\sqrt{a^2+b^2}$, $A=\begin{bmatrix}r & 0\\ 0 & r\end{bmatrix}\begin{bmatrix}\frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}}\\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}}\end{bmatrix}$

$\Rightarrow \underbrace{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2}_{\cos\alpha}+\underbrace{\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2}_{\sin\alpha}=1$

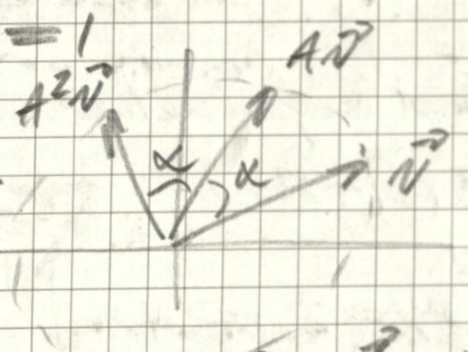
$\Rightarrow A=\begin{bmatrix}r & 0\\ 0 & r\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$



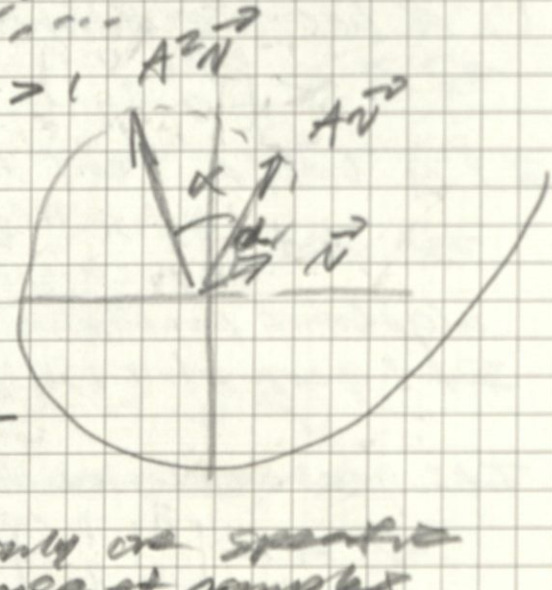
We want to draw $\vec{v}, A\vec{v}, A^2\vec{v}, \ldots$

1) $r=1$

circle



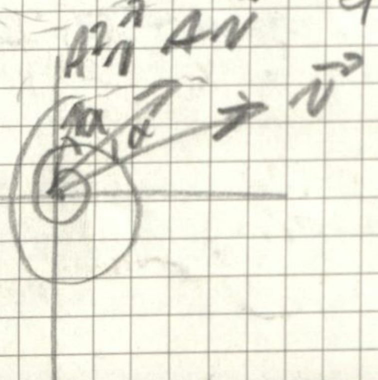
2) $r>1$



spiral out

2) $r<1$

spiral in



* only one specific case of complex eigenvalue transformations

Ex- $A = \begin{bmatrix} 1 & -6 \\ 3 & 7 \end{bmatrix} \Rightarrow$ Char. eqn. $\begin{vmatrix} 1-\lambda & -6 \\ 3 & 7-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda-1)(\lambda-7)+18=0 \Rightarrow \lambda^2-8\lambda+25=0$

$\Rightarrow \underbrace{(\lambda^2-8\lambda+16)}+9=0$

$$(\lambda-4)^2+3^2=0 \Rightarrow \lambda_{1,2}=4\pm 3i$$

$\lambda_1 = 4-3i$: $\begin{bmatrix} -3+3i & -6 \\ 3 & 3+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1+i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

compute: $P^{-1}AP = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} -3 & -7 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

* other eigenvector will produce a conjugate matrix

Suppose, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has two complex eigenvalues $\lambda_{1,2} = a \pm bi$

1) Find an eigenvector corresponding to $a-bi$

$\vec{v} = Re\,\vec{v} + i\,Im\,\vec{v} \rightsquigarrow P = [Re\,\vec{v} \;\; Im\,\vec{v}]$

$\Rightarrow P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Qualitatively:

1) $\sqrt{a^2+b^2} > 1$ $A\vec{x}, A^2\vec{x}, \ldots$ points on out spiral

2) $\sqrt{a^2+b^2} < 1$ $A\vec{x}, A^2\vec{x}, \ldots$ points on in spiral

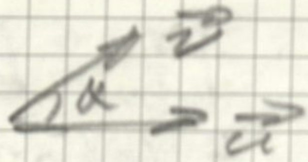
3 $\sqrt{a^2+b^2} = 1$ $A\vec{x}, A^2\vec{x}, \ldots$ points on an ellipse

diagonalization → basis is eigenvectors

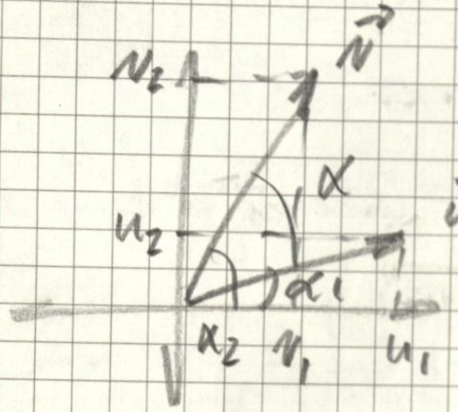
similarity → basis is real and imaginary

Inner Products:

Recall - In $\mathbb{R}^2$, $\vec{u}\cdot\vec{v}$ is defined for



as $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\alpha$



$\alpha = \alpha_2 - \alpha_1$

$\cos(\alpha_2-\alpha_1) = \cos\alpha_2\cos\alpha_1 + \sin\alpha_2\sin\alpha_1$

$$= \frac{v_1}{\sqrt{v_1^2+v_2^2}}\frac{u_1}{\sqrt{u_1^2+u_2^2}} + \frac{v_2}{\sqrt{v_1^2+v_2^2}}\frac{u_2}{\sqrt{u_1^2+u_2^2}}$$

$|\vec{u}| = \sqrt{u_1^2+u_2^2}$

$|\vec{v}| = \sqrt{v_1^2+v_2^2}$

$\Rightarrow \vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2$

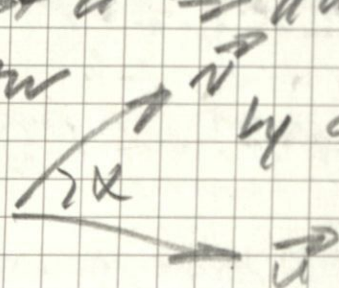
Properties

1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
3) $(c\vec{u})\cdot\vec{w} = c(\vec{u}\cdot\vec{w})$
4) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0 \Leftrightarrow \vec{u} = 0$

* for any vector, an operation with these properties is called a $\cdot$-product or inner product

Can also define:

1) length of $\vec{u} \Rightarrow \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$
2) angle btw



by $\cos\alpha = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$

Lecture Notes: 5.22.24

$\mathbb{R}^n$. Notion of a dot product (inner product)

Def. A dot product is an operation that takes any two vectors $\vec{u}, \vec{v}$ and computes a number $\vec{u}\cdot\vec{v}$

s.t. 1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

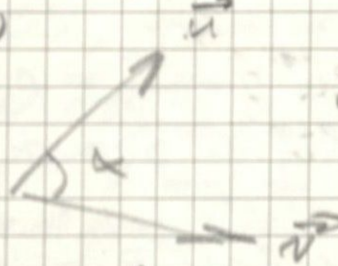
2) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

3) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

4) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0 \iff \vec{u} = 0$

Def. 1) Length of $\vec{u}$, $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

2)



$$\cos\alpha = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

Need to Check:

$|\cos\alpha| \leq 1 \iff \dfrac{(\vec{u}\cdot\vec{v})^2}{\|\vec{u}\|^2\|\vec{v}\|^2} \leq 1$

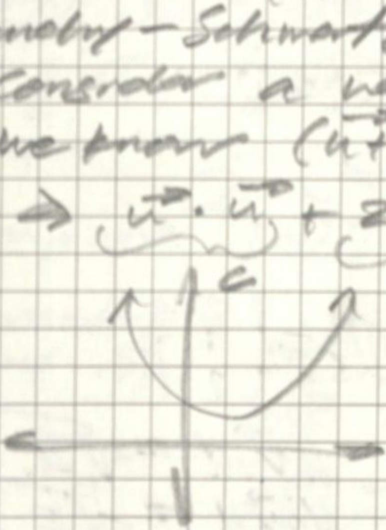
$(\vec{u}\cdot\vec{v})^2 \leq (\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v})$

* Cauchy–Schwarz–Ostrogradsky

Consider a vector $\vec{u} + x\vec{v}$

we know $(\vec{u}+x\vec{v})(\vec{u}+x\vec{v}) \geq 0$

$$\Rightarrow \underbrace{\vec{u}\cdot\vec{u}}_{c} + \underbrace{2(\vec{u}\cdot\vec{v})x}_{b} + \underbrace{(\vec{v}\cdot\vec{v})x^2}_{a} \geq 0$$



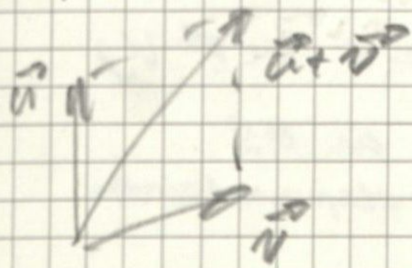
discriminant $= b^2 - 4ac \leq 0$

$\Rightarrow 4(\vec{u}\cdot\vec{v})^2 - 4(\vec{v}\cdot\vec{v})\cdot(\vec{u}\cdot\vec{u}) \leq 0$

$\Rightarrow (\vec{u}\cdot\vec{v})^2 \leq (\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v})$

Also Need:

$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ ✓



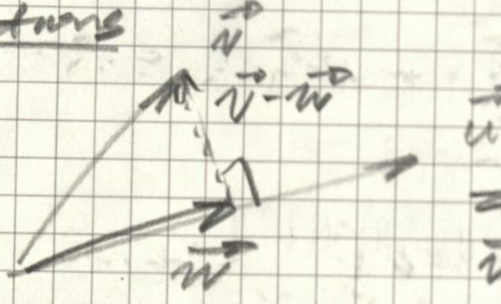
3) $\vec{u} \perp \vec{v} \iff \vec{u}\cdot\vec{v} = 0$

Main example of a dot product in $\mathbb{R}^n$:

Def. For $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = [u_1\ u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$
$$= \vec{u}^T \cdot \vec{v}$$

Projections



1) $\vec{w} = c\vec{u}$

2) $(\vec{v} - \vec{w}) \cdot \vec{u} = 0$

$\Rightarrow (\vec{v} - c\vec{u}) \cdot \vec{u} = 0$

$\vec{v} \cdot \vec{u} - c(\vec{u} \cdot \vec{u}) = 0$

$\Rightarrow c = \dfrac{\vec{v} \cdot \vec{u}}{(\vec{u} \cdot \vec{u})}$

$$\Rightarrow \operatorname{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Orthogonal and Orthonormal Systems of Vectors:

Ex: $\mathbb{R}^n$ has standard basis $\vec{e}_1, \dots, \vec{e}_n$

$\vec{e}_i \cdot \vec{e}_j = 0 \Leftrightarrow \vec{e}_i \perp \vec{e}_j \quad i \neq j$

$\vec{e}_i \cdot \vec{e}_i = 1 \Leftrightarrow \|\vec{e}_i\| = 1$

Ex: $\mathbb{R}^3$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = 0$ $\quad \|\vec{v}_1\| = \sqrt{3}$

$\vec{v}_2 \cdot \vec{v}_3 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$ $\quad \|\vec{v}_2\| = \sqrt{2}$

$\|\vec{v}_3\| = \sqrt{6}$

Def. $\{\vec{v}_1, \dots, \vec{v}_k\} \in \mathbb{R}^n$ is an orthogonal system if $\vec{v}_i \cdot \vec{v}_j = 0 \ \forall i \neq j$

Property. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an OS then $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent

Why? Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = 0$

- multiply this eq-n by $\vec{v}_1$

$\vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k) = 0$

$c_1 \underbrace{(\vec{v}_1 \cdot \vec{v}_1)}_{0} = 0 \Rightarrow c_1 = 0$

Similarly, $c_2, c_3, \ldots, c_k = 0$

so $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ independent

Consequence - If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \in \mathbb{R}^n$ is an OS, then it is a basis

Ex - $W \in \mathbb{R}^3$

$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x+y+z=0 \right\}$ → Kernel

$\Updownarrow$

$[1 \ 1 \ 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Pick $\underbrace{}_{\#\text{ of free vars}}$:

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \in W \Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0 \Leftrightarrow \vec{v}_1 \perp \vec{v}_2$

$\Rightarrow \vec{v}_1, \vec{v}_2$ are a basis in W

Let $\underbrace{\{\vec{v}_1, \ldots, \vec{v}_k\}}_{\mathcal{B}}$ be an Orthogonal Basis in W

Let $\forall \vec{w} \in W$

Compute $[\vec{w}]_{\mathcal{B}} \Leftrightarrow \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$

- multiply by $\vec{v}_1$ $\Rightarrow \vec{v}_1 \cdot \vec{w} = c_1(\vec{v}_1 \cdot \vec{v}_1)$

$\Rightarrow c_1 = \dfrac{\vec{v}_1 \cdot \vec{w}}{(\vec{v}_1 \cdot \vec{v}_1)}$

$\Rightarrow c_i = \dfrac{\vec{v}_i \cdot \vec{w}}{(\vec{v}_i \cdot \vec{v}_i)}$

Orthonormal Systems:

$\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ is an orthonormal system (basis) if, in addition to $\vec{v_i} \cdot \vec{v_j} = 0 \ \forall i \neq j$, we have $\vec{v_i} \cdot \vec{v_i} = 1 \iff \|\vec{v_i}\| = 1$

Let $\mathcal{B} = \{\vec{q_1}, \vec{q_2}, \dots, \vec{q_k}\}$ is an ONB $\in W$

then $\forall \vec{w} \in W$ can be written as:

$\vec{w} = (\vec{w} \cdot \vec{q_1})\vec{q_1} + (\vec{w} \cdot \vec{q_2})\vec{q_2} + \dots + (\vec{w} \cdot \vec{q_k})\vec{q_k}$

Let $\vec{q_1}, \dots, \vec{q_n}$ be an ONB $\in \mathbb{R}^n$

define $Q = [\vec{q_1} \ \vec{q_2} \ \dots \ \vec{q_n}]$

1) Q is invertable

2) $Q^{-1} = Q^T$

3) $\|Q\vec{x}\| = \|\vec{x}\|$

4) $\vec{x}, \vec{y} \Rightarrow (Q\vec{x}) \cdot (Q\vec{y}) = \vec{x} \cdot \vec{y}$

Such Q is called an orthogonal matrix

5) $\det Q = \pm 1$

Ex:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det = 1$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det = -1$

Lecture Notes: 3.25.24

Recall: $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n$, $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow u \perp \vec{v}$

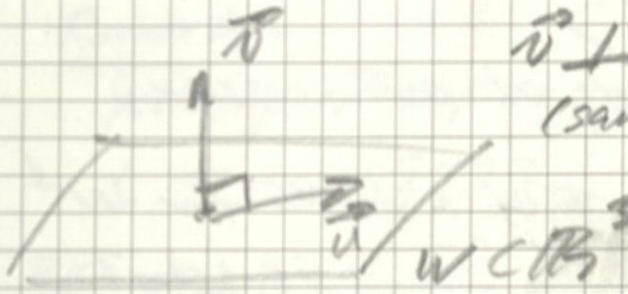
Def. An orthogonal basis for W is a basis which is an orthogonal set

Def. A set is orthogonal if every pair of two vectors are orthogonal

Ex. In $\mathbb{R}^3$, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an ortho. set

Property. An orthogonal set of nonzero vectors is always a linearly independent set

Orthogonal Complement:



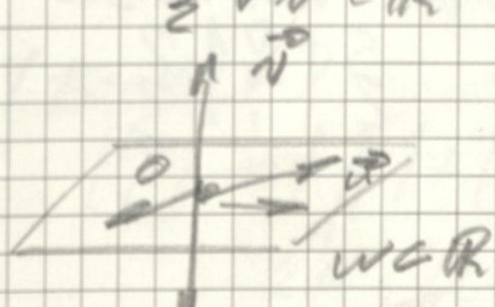
$\vec{v} \perp W \Rightarrow \vec{v} \in W^\perp$

(same with any scalar multiple of $\vec{v}$)

Def. If $W \subset \mathbb{R}^n$ is a subspace and $\vec{z} \in \mathbb{R}^n$ is orthogonal to W if $\vec{z}$ is orthogonal to each vector in W

Def. The set of all vectors in $\mathbb{R}^n$ orthogonal to W is "orthogonal complement of W" $\rightarrow W^\perp$

$\{\forall \vec{v} \in \mathbb{R}^n : \vec{v} \perp W \in \mathbb{R}^n\} = W^\perp$



W: plane

$L = W^\perp$: line

$L^\perp = W = (W^\perp)^\perp = W$

Properties:

1) $(W^\perp)^\perp = W$

2) $W^\perp$ is a subspace of $\mathbb{R}^n$

Thm. $\dim W + \dim W^\perp = n$ ($W \subset \mathbb{R}^n$ subspace)

For $A \in M_{mn}$,

1) $(\text{row } A)^\perp = \text{null}(A)$

2) $(\text{col} A)^\perp = \text{null}(A^T)$

Ex: ① $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}_{2\times 3}$

row(A) null(A)

$\text{null}(A)$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ st. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\Rightarrow \begin{cases} 1\cdot x + 2\cdot y + 3\cdot z = 0 \rightsquigarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \\ 1\cdot x + 0\cdot y + 1\cdot z = 0 \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \end{cases}$

② $\text{col}(A) = \text{row}(A^T)$

$\text{row}(A^T) \perp \text{null}(A^T)$

$\Rightarrow \text{col}(A) \perp \text{null}(A^T)$

(orthogonal complements of each other)

Ex: $W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$. Find a basis for $W^\perp$

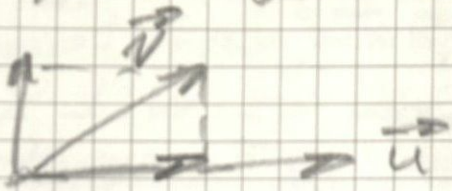
$W = \text{col}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\right) \Leftrightarrow W^\perp = \text{null}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}^T\right)$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 + x_3 = 0, \; x_3 = s \end{matrix}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} \overset{\text{free}}{=} s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow W^\perp$ has bases

$\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

Orthogonal Projection onto a line:



For $\vec{u}, \vec{v} \in \mathbb{R}^n$, we can decompose $\vec{v} = \hat{v} + \vec{z}$

$(\parallel \vec{u})$ $(\perp \vec{u})$

$\hat{v} = \alpha \cdot \vec{u}$

$\vec{v} = \alpha \vec{u} + \vec{z}$ $(\cdot \vec{u})$ $\quad \vec{v} \cdot \vec{u} = \alpha(\vec{u} \cdot \vec{u}) + \underbrace{(\vec{z} \cdot \vec{u})}_{0}$

$\Rightarrow \alpha = \dfrac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

$\hat{v} = \left(\dfrac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \text{proj}_{\vec{u}} \vec{v}$ (proj of $\vec{v}$ onto $\vec{u}$)

$\vec{z} = \vec{v} - \hat{v} = \text{perp}_{\vec{u}} \vec{v} = \vec{v} - \left(\dfrac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$

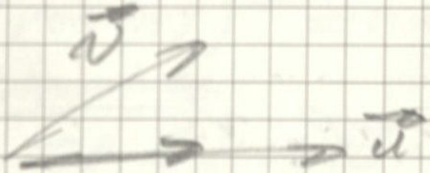
$\vec{v} = \text{proj}_{\vec{u}} \vec{v} + \text{perp}_{\vec{u}} \vec{v}$

__Remarks:__

$\text{proj}_{\vec{u}} \vec{v} = \text{proj}_{c\vec{u}} \vec{v} = \text{proj}_L \vec{v}$

$L = \text{span}\{\vec{u}\}$

__Ex:__ $\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. __Q:__ Find $\text{proj}_{\vec{u}} \vec{v}$, $\text{perp}_{\vec{u}} \vec{v}$



$\text{proj}_{\vec{u}} \vec{v} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$

$= \dfrac{28+12}{16+4}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\text{perp}_{\vec{u}} \vec{v} = \vec{v} - \text{proj}_{\vec{u}} \vec{v}$

$= \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(__Finally:__)

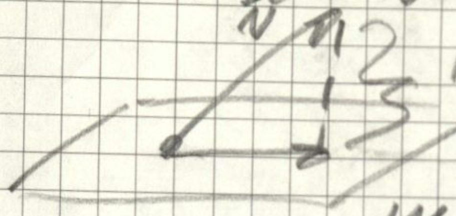
$\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

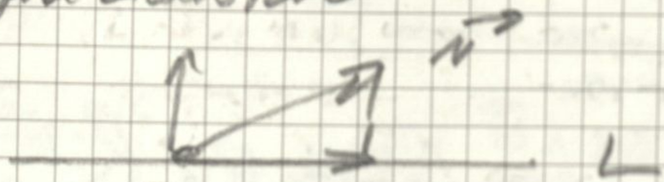
$\parallel$

$2\vec{u}$ $\quad (\perp \vec{u})$

$(c\vec{u})$

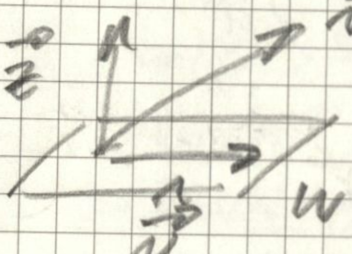
Q: $(\vec{u}, \vec{v})$. Find the distance from $\vec{v}$ to $L = \text{span}\{\vec{u}\}$





$\text{dist}(\vec{v}, L) = \|\text{perp}_L \vec{v}\| = \|\vec{v} - \text{proj}_L \vec{v}\| = \left\|\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\| = \sqrt{5}$

<u>Orthogonal Decomposition Thm</u>:



Let $W \subset \mathbb{R}^n$ be a subspace and $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ be an <u>orthogonal basis</u> for W

Then, $\forall \vec{v} \in \mathbb{R}^n$, there are <u>unique</u> vectors $\hat{\vec{v}} \in W$, $\vec{z} \in W^\perp$ st. $\vec{v} = \underset{(W)}{\hat{\vec{v}}} + \underset{(W^\perp)}{\vec{z}}$

$$\underset{\text{proj}_W \vec{v}}{\overset{\shortparallel}{\hat{\vec{v}}}} = \left(\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2 + \cdots + \left(\frac{\vec{v}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p$$

$\vec{z} = \vec{v} - \hat{\vec{v}} = \vec{v} - \text{proj}_W \vec{v} = \text{perp}_W \vec{v}$

$\hat{\vec{v}}$: the projection of $\vec{v}$ onto W

$\vec{z}$: the component of $\vec{v}$ orthogonal to W

<u>Remark</u>:

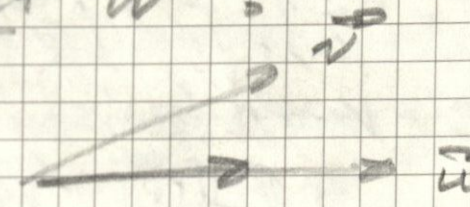
1) $\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \text{proj}_{\vec{u}_2} \vec{v} + \dots + \text{proj}_{\vec{u}_p} \vec{v}$

2) If $\vec{v} \in W$, then $\text{proj}_W \vec{v} = \vec{v}$ and $\text{perp}_W \vec{v} = \vec{0}$

Lecture Notes: 5.27.24

Recap: orthogonal complement $W^\perp$:

$x \perp W \iff x \in W^\perp$

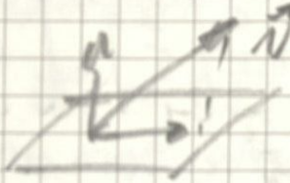


orthogonal projection:

$\vec{v} = \text{proj}_{\vec{u}} \vec{v} + \text{perp}_{\vec{u}} \vec{v}$

$\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$

Generalize to $\forall W$:



$\vec{v} = \underbrace{\text{proj}_W \vec{v}}_{\in W} + \underbrace{\text{perp}_W \vec{v}}_{\in W^\perp}$

Assume W has a orthonormal basis $\{\vec{u}_1, \dots \vec{u}_p\}$

1) $\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \dots + \text{proj}_{\vec{u}_p} \vec{v}$

$= \left(\frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \dots + \left(\frac{\vec{v} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p$

Orthogonal Complement:

$[\text{col}(A)]^\perp = \text{null}(A^T)$

$[\text{row}(A)]^\perp = \text{null}(A)$

Properties:

2) If $\vec{v} \in W$, then $\text{proj}_W \vec{v} = \vec{v}$

3) $W = \mathbb{R}^n$. assume $\{\vec{u}_1, \dots \vec{u}_n\}$ is an OB of $\mathbb{R}^n$

$\vec{v} = \text{proj}_W \vec{v} = \left(\frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \dots + \left(\frac{\vec{v} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n} \right) \vec{u}_n$

$\rightarrow [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \\ \vdots \\ \frac{\vec{v} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n} \end{bmatrix}$. If $\mathcal{B}$ is an OB, we have a simple way to find $[\vec{v}]_{\mathcal{B}}$

ex- $W = \mathbb{R}^2$, $\vec{u}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$. $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$.

$\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$. Find $[\vec{v}]_{\mathcal{B}}$

Sol- Verify $\mathcal{B}$ is an OB. $\vec{u}_1 \cdot \vec{u}_2 = 0$ ✓

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \dfrac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} \\ \dfrac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2} \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$$

Ex: $\vec{u}_1 = \begin{bmatrix}2\\5\\-1\end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix}-2\\1\\1\end{bmatrix}$. $W = \text{span}\left\{\begin{bmatrix}2\\5\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}$

$\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Find $\text{proj}_W \vec{v}$, $\text{perp}_W \vec{v}$

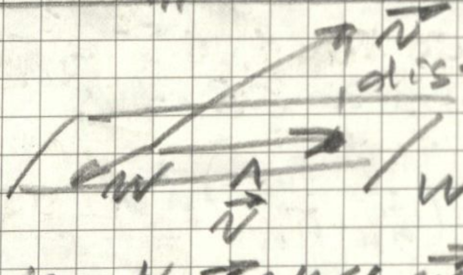
Sol. $\vec{u}_1 \cdot \vec{u}_2 = -4+5-1 = 0$

$$\text{proj}_W \vec{v} = \left(\frac{\vec{v}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{v}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$$

$$= \frac{2+10-3}{4+25+1}\begin{bmatrix}2\\5\\-1\end{bmatrix} + \frac{-2+2+3}{4+1+1}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix} = \begin{bmatrix}7/5\\0\\14/5\end{bmatrix}$$

Best Approximation Thm:



Let $W \subset \mathbb{R}^n$, $\vec{v} \in \mathbb{R}^n$, $\hat{\vec{v}} = \text{proj}_W \vec{v}$

Then $\hat{\vec{v}}$ is the closest point on W to $\vec{v}$

i.e. $\forall \vec{w} \in W: \vec{w} \neq \hat{\vec{v}} \Rightarrow \|\vec{v} - \vec{w}\| > \|\vec{v} - \hat{\vec{v}}\|$

uniqueness

$\hat{\vec{v}}$ is the best approximation of $\vec{v}$ by an element of W

& $\|\vec{v} - \hat{\vec{v}}\|$ is the "error of approximation"

eg: $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$ (prev ex)

$$\text{dist}(\vec{v}, W) = \|\vec{v} - \text{proj}_W \vec{v}\| = \|\text{perp}_W \vec{v}\| = \left\|\begin{bmatrix}7/5\\0\\14/5\end{bmatrix}\right\| = \frac{7}{5}\left\|\begin{bmatrix}1\\0\\2\end{bmatrix}\right\|$$

Def. A set of vectors in $\mathbb{R}^n$ is an orthonormal set if it is an orthogonal set of unit vectors

* orthonormal basis if ① basis ② orthonormal set

* To verify a basis is an orthonormal basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_p\}$. $\vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

ex. $\{\vec{e}_1, \vec{e}_2, \dots \vec{e}_n\}$. the standard basis is an orthonormal basis

* orthonormal : O/N

ex. $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis but not O/N

normalization : $\vec{v}_1 = \dfrac{\vec{u}_1}{\|\vec{u}_1\|}$, $\vec{v}_2 = \dfrac{\vec{u}_2}{\|\vec{u}_2\|}$

$\{\vec{v}_1, \vec{v}_2\}$ gives an O/N.

$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thm. An $m \times n$ matrix U has O/N cols $\Leftrightarrow U^T U = I$

ex. $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \Leftrightarrow U^T U = I$ ($U^T = U$, $\Rightarrow U^2 = I \Rightarrow U = U^{-1}$)

Thm. Let $U \in M_{mn}$ with O/N cols. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$

Properties:

1) $\|U \cdot \vec{x}\| = \|\vec{x}\|$

2) $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$

3) $U\vec{x} \perp U\vec{y} \Leftrightarrow \vec{x} \perp \vec{y}$

* U preserves length, dot product, orthogonality

ex. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = U$. U has O/N columns b/c $\sin^2\theta + \cos^2\theta = 1$

Geometrically : U preserves length and angle

Thm : If $\{\vec{u}_1, \dots \vec{u}_p\}$ is O/N set. Then :

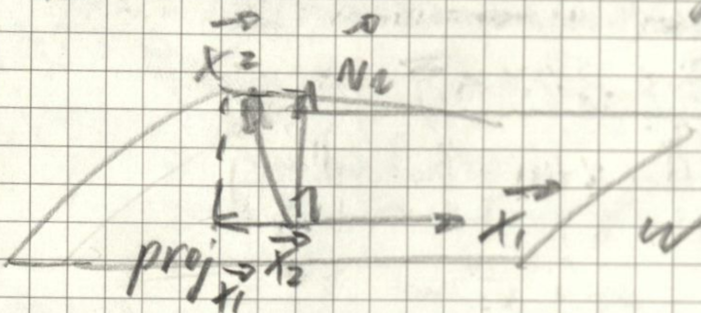
1) $\text{proj}_W \vec{v} = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{v} \cdot \vec{u}_p)\vec{u}_p$

2) If $U = [\vec{u}_1, \dots \vec{u}_p]_{n \times p}$ then $\text{proj}_W \vec{v} = U \cdot U^T \cdot \vec{v}$

The Gram-Schmidt Process: 5-31-24

$W = \text{span}\{\vec{x}_1, \vec{x}_2\}$, $\vec{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$

Goal: Construct an orthogonal basis for W



$$\vec{v}_2 = \text{perp}_{\vec{x}_1}\vec{x}_2 = \vec{x}_2 - \text{proj}_{\vec{x}_1}\vec{x}_2$$

$$= \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\right)\vec{x}_1$$

Thm. Let $\{\vec{x}_1, \dots \vec{x}_k\}$ be a basis for $W \subseteq \mathbb{R}^n$

$\vec{v}_1 = \vec{x}_1$, $\qquad W_1 = \text{span}\{\vec{x}_1\}$

$\vec{v}_2 = \text{perp}_{W_1}\vec{x}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_1}\vec{x}_2 \quad W_2 = \text{span}\{\vec{x}_1, \vec{x}_2\}$

$\vdots$

$$\vec{v}_k = \vec{x}_k - \left(\frac{\vec{x}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_k \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 - \dots$$

$$\| \qquad - \left(\frac{\vec{x}_k - \vec{v}_{k-1}}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}}\right)\vec{v}_{k-1}, \quad W_k = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\}$$

$\text{perp}_{W_{k-1}}\vec{x}_k$

* Every subspace of $\mathbb{R}^n$ has an orthogonal basis

QR Factorization:

A_{mn} where $m \geq n$.

Applying Gram-Schmidt gives factorization of A into O/N matrix Q and upper triangular matrix R

$$A = [\underbrace{\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n}_{\text{lin ind}}] = [\underbrace{\vec{q}_1\ \vec{q}_2\ \dots\ \vec{q}_n}_{\text{O/N}}]\begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn}\end{bmatrix} = QR$$

w/ Gram-Schmidt

Since Q is O/N, $Q^TQ = I$

$\therefore A = QR \iff R = Q^TA$

|Independent Notes| $q\downarrow$

Least Squares Approximation:

Find some equation $\mathcal{P}_n$ as a "line of best fit" through points $(x_1, y_1), \ldots, (x_k, y_k)$

Note: $\mathcal{P}$ is a linear space, but in particular $\mathcal{P}_1 : ax+b=y$

Suppose: data points $(1,2), (2,2), (3,4)$

$$\Rightarrow \begin{cases} a+b=2 \\ 2a+b=2 \\ 3a+b=4 \end{cases} \Longleftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$A \cdot \vec{x} = \vec{b}$$

This system in particular is inconsistent, so now look to minimize error instead of finding exact solution

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \quad \|\vec{e}\| = |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}$$

↳ error vector ↳ least squares error

$\varepsilon_1 = 2-(a+b\cdot 1)$, $\varepsilon_2 = 2-(a+b\cdot 2)$, $\varepsilon_3 = 4-(a+b\cdot 3)$

Definition: A is $m \times n$, $b \in \mathbb{R}^m$, least squares solution of $A\vec{x}=\vec{b}$ is a vector $\hat{x} \in \mathbb{R}^n$ st.
$\|\vec{b} - A\hat{x}\| \le \|\vec{b} - A\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$

Solution: Any vector in form $A\vec{x} \in col(A)$ as $x \in row(A)$ varies over all vectors in $\mathbb{R}^n$

$\therefore A\hat{x} = \vec{b} \Longleftrightarrow \|\vec{b} - q\| \le \|\vec{b} - \vec{y}\| \quad \forall \vec{y} \in col(A)$

ie. "best approximation" in $col(A)$ to $\vec{b}$

$\therefore A\hat{x} = proj_{col(A)} \vec{b} \Longleftrightarrow \vec{b} - A\hat{x} = \vec{b} - proj_{col(A)} \vec{b}$
$= perp_{col(A)} \vec{b} \perp col(A) \therefore \vec{b} - A\hat{x} \in col(A)^{\perp} = null(A^T)$

$\therefore A^T(\vec{b} - A\hat{x}) = \vec{0} \Longleftrightarrow A^TA\hat{x} = A^T\vec{b}$ (normal eqns)

Thm. Let A be $m \times n$ and $\vec{b} \in \mathbb{R}^m$. $A\vec{x} = \vec{b}$ always has at least 1 solution $\hat{x}$ st.

1) $\hat{x}$ is a LSS of $A\vec{x} = \vec{b} \Longleftrightarrow \hat{x}$ is a solution of $A^TA\hat{x} = A^T\vec{b}$

2) A has linearly independent columns $\Longleftrightarrow A^TA$ is invertible, which means LSS is unique:
$\hat{x} = (A^TA)^{-1}A^T\vec{b}$

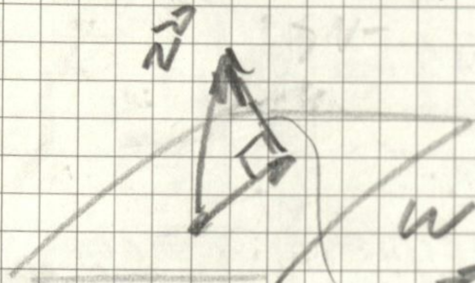
3) If $A = QR$,
$\hat{x} = R^{-1}Q^T\vec{b}$

Lecture Notes 4.3.24

Orthonormal Basis and Orthogonal Projections

W - Subspace in $\mathbb{R}^n$, Basis $\vec{v}_1, \dots, \vec{v}_k$ in W which is orthogonal, i.e. $\vec{v}_i \cdot \vec{v}_j = 0$ ($\vec{v}_i \perp \vec{v}_j$) ($i \neq j$)

Want to find $\mathrm{proj}_W \vec{v}$?



$$\mathrm{proj}_W \vec{v} = \frac{(\vec{v}_1 \cdot \vec{v})}{(\vec{v}_1 \cdot \vec{v}_1)}\vec{v}_1 + \frac{(\vec{v}_2 \cdot \vec{v})}{(\vec{v}_2 \cdot \vec{v}_2)}\vec{v}_2 + \dots + \frac{(\vec{v}_k \cdot \vec{v})}{(\vec{v}_k \cdot \vec{v}_k)}\vec{v}_k$$

$\Rightarrow \vec{v} - \mathrm{proj}_W \vec{v} = \mathrm{perp}_W \vec{v}$

$$\Rightarrow \vec{v} = \underbrace{\mathrm{proj}_W \vec{v}}_{\in W} + \underbrace{\mathrm{perp}_W \vec{v}}_{\in W^\perp}$$

Projection is nicer if a basis in W is orthonormal

$\vec{q}_1, \dots, \vec{q}_k$ - Basis in W

$\vec{q}_i \perp \vec{q}_j$ $\oplus$ $\vec{q}_i \cdot \vec{q}_i = 1$

$\Longleftrightarrow \|\vec{q}_i\| = 1$

$$\mathrm{proj}_W v = (\vec{q}_1 \cdot v)\vec{q}_1 + (\vec{q}_2 \cdot v)\vec{q}_2 + \dots + (\vec{q}_k \cdot v)\vec{q}_k$$

$$= \underbrace{\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \dots & \vec{q}_k \end{bmatrix}}_{n\{\underbrace{Q}_{k}} \begin{pmatrix} \vec{q}_1 \cdot v \\ \vec{q}_2 \cdot v \\ \vdots \\ \vec{q}_k \cdot v \end{pmatrix} = \begin{bmatrix} \vec{q}_1 & \dots & \vec{q}_k \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_k^{\,T} \end{bmatrix} \vec{v}$$

$\Rightarrow QQ^T\vec{v}$

Here Q has orthonormal columns, so $Q^TQ = I_k$

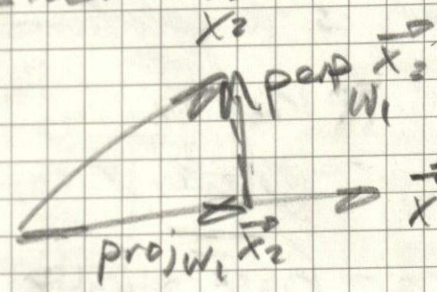
How to construct Orthogonal Basis:

Take any basis in W, $\vec{x}_1, \ldots, \vec{x}_k$
Want to produce $\vec{v}_1, \ldots, \vec{v}_k$ which is an OB

1) $\vec{v}_1 = \vec{x}_1$, $W_1 = \text{span}\{\vec{v}_1\} = \text{span}\{\vec{x}_1\}$

2) <u>Idea</u>: Take $\vec{x}_2$ and consider

$$\vec{x}_2 - \text{proj}_{W_1}\vec{x}_2 = \text{perp}_{W_1}\vec{x}_2$$

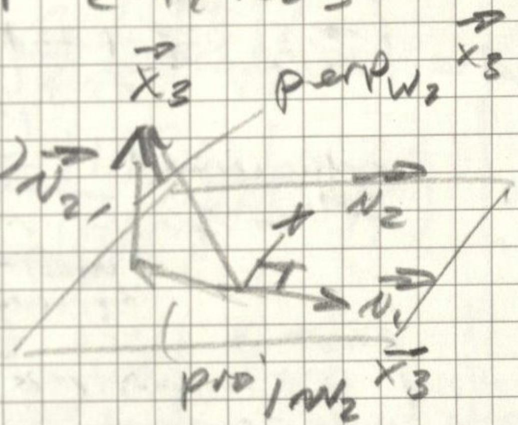


$$\Rightarrow \vec{x}_2 - \frac{(\vec{v}_1 \cdot \vec{x}_2)}{(\vec{v}_1 \cdot \vec{v}_1)}\vec{v}_1,$$

$W_2 = \text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{x}_1, \vec{x}_2\}$

3) $\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2}\vec{x}_3$

$$= \vec{x}_3 - \frac{(\vec{v}_1 \cdot \vec{x}_3)}{(\vec{v}_1 \cdot \vec{v}_1)}\vec{v}_1 - \frac{(\vec{v}_2 \cdot \vec{x}_3)}{(\vec{v}_2 \cdot \vec{v}_2)}\vec{v}_2,$$



$W_3 = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$
$= \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

$\vdots$

k) $\vec{v}_k = \vec{x}_k - \text{proj}_{W_{k-1}}\vec{x}_k$, $W_{k-1} = \text{span}\{\vec{v}_1, \ldots, \vec{v}_{k-1}\}$

$$\Rightarrow \vec{x}_k - \frac{(\vec{v}_1 \cdot \vec{x}_k)}{(\vec{v}_1 \cdot \vec{v}_1)}\vec{v}_1 - \frac{(\vec{v}_2 \cdot \vec{x}_k)}{(\vec{v}_2 \cdot \vec{v}_2)}\vec{v}_2 - \ldots - \frac{(\vec{v}_{k-1} \cdot \vec{x}_k)}{(\vec{v}_{k-1} \cdot \vec{v}_{k-1})}\vec{v}_{k-1}$$

Called
* Gram-Schmidt Orthogonalization *

To make the resulting basis orthonormal define

$$\vec{q}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1,\ \vec{q}_2 = \frac{1}{\|\vec{v}_2\|}\vec{v}_2, \ldots, \vec{q}_k = \frac{1}{\|\vec{v}_k\|}\vec{v}_k$$

If original $\vec{x}_1, \ldots, \vec{x}_k$ were arranged as columns of a matrix

$A = [\vec{x}_1 \ldots \vec{x}_k]$ and the eventual $\vec{q}_1, \ldots \vec{q}_k$

as columns of $Q = [\vec{q}_1, \ldots, \vec{q}_k]$

What is the relation between A and Q?

$\vec{x}_1 = r_{11} \cdot \vec{q}_1$

$\vec{x}_2 = r_{12}\vec{q}_1 + r_{22}\vec{q}_2$, since $\vec{x}_2$ lives in $W_2 = \text{span}\{\vec{q}_1, \vec{q}_2\}$

$\vec{x}_3 = r_{13}\vec{q}_1 + r_{23}\vec{q}_2 + r_{33}\vec{q}_3$

$\vdots$

$$\Rightarrow A = [\vec{x}_1 \ldots \vec{x}_k] = \underbrace{[\vec{q}_1 \ldots \vec{q}_k]}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1k} \\ 0 & r_{22} & r_{23} & \cdots & r_{2k} \\ 0 & 0 & r_{33} & \cdots & r_{3k} \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & 0 & \cdots & r_{kk} \end{bmatrix}}_{R}$$

orthonormal columns $\longrightarrow Q$

(upper) triangular $\longrightarrow R$

Thm: Any $n \times k$ matrix with linearly independent columns can be written as:

$$A = QR$$

orthonormal columns (Q), upper triangular (R)

* Called QR Factorization

Ex: Basis in $\mathbb{R}_3$: $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

1) Apply G-S to produce orthonormal basis
2) Find QR Factorization of $A \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ -1 & 3 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \vec{x}_2 - \dfrac{(\vec{v}_1 \cdot \vec{x}_2)}{(\vec{v}_1 \cdot \vec{v}_1)} \vec{v}_1 = \vec{x}_2$

$\vec{v}_3 = \vec{x}_3 - \dfrac{(\vec{v}_1 \cdot \vec{x}_3)}{(\vec{v}_1 \cdot \vec{v}_1)} \vec{v}_1 - \dfrac{(\vec{v}_2 \cdot \vec{x}_3)}{(\vec{v}_2 \cdot \vec{v}_2)} \vec{v}_2 = \vec{x}_3 - \frac{5}{3}\vec{v}_1 - \vec{v}_2$

$\Rightarrow \vec{v}_3 = \frac{1}{3}\begin{bmatrix}4\\2\\-2\end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = 0, \ \vec{v}_1 \cdot \vec{v}_3 = 0, \ \vec{v}_2 \cdot \vec{v}_3 = 0$ ✓

$\Rightarrow \vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}, \vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\-1\end{bmatrix}$

$A = Q \cdot R \Rightarrow R = Q^T A$ ✓

$$\Rightarrow \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6}\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\ -1 & 3 & 2\\ 1 & 3 & 4\end{bmatrix} = \begin{bmatrix}\cdot & \cdot & \cdot\\ 0 & \cdot & \cdot\\ 0 & 0 & \cdot\end{bmatrix}$$

Ex. $W_2 = \text{span}\left\{\begin{bmatrix}2\\-4\end{bmatrix}\right\} = \text{span}\left\{\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}\right\}$

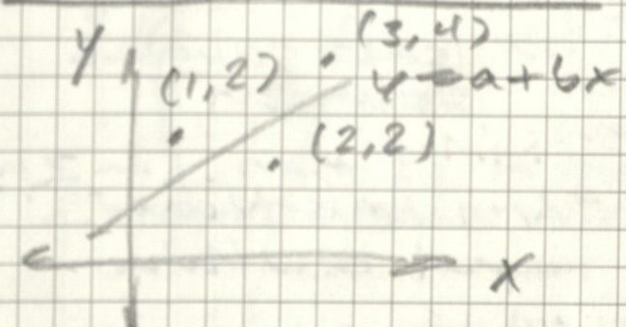
$Q = \begin{bmatrix}1/\sqrt{5}\\-2/\sqrt{5}\end{bmatrix}, P = QQ^T = \begin{bmatrix}1/\sqrt{5}\\-2/\sqrt{5}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}}\end{bmatrix} = \begin{bmatrix}1/5 & -2/5\\ -2/5 & 4/5\end{bmatrix}$

OR

$\text{proj}_{W_2}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{x_1 - 2x_2}{5}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}\frac{x_1-2x_2}{5}\\ \frac{-2x_1+4x_2}{5}\end{bmatrix}$

Lecture Notes: 4.5.24

Least Squares Solutions:



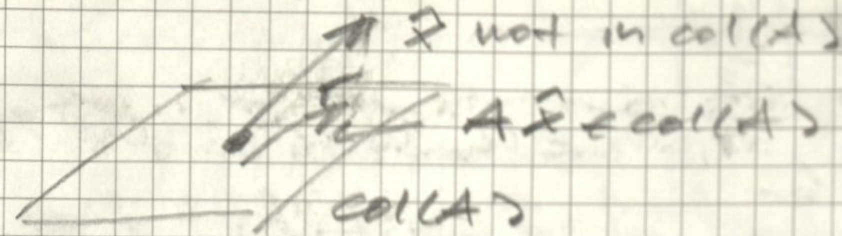
$a + b \cdot 1 = 2$
$a + b \cdot 2 = 2$
$a + b \cdot 3 = 4$

$\Leftrightarrow \begin{bmatrix}1 & 1\\1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2\\2\\4\end{bmatrix}$

$\Leftrightarrow A\vec{x} = \vec{b}$

* No exact solution, but we can try to find $\vec{x}$ that $A\vec{x}$ is the closest to $\vec{b}$

This means that $\|A\vec{x} - \vec{b}\|$ is the smallest possible



Replace question/equation with

$$A\hat{x} = \text{proj}_{\text{col}(A)} \vec{b}$$

Recall - fundamental subspaces associated with a matrix A, A is $m \times n$

$$\text{col}(A),\ \text{row}(A),\ \text{null}(A),\ \text{null}(A^T)$$

Any vector $\vec{x}$ which is orthogonal to row(A) satisfies $A \cdot \vec{x} = 0$

$\Rightarrow \vec{x} \in \text{null}(A)$

$$[\text{row}(A)]^{\perp} = \text{null}(A)$$

Similarly -

$$[\text{col}(A)]^{\perp} = \text{null}(A^T)$$

$\Rightarrow (\vec{b} - \text{proj}_{\text{col}(A)} \vec{b}) \perp \text{col}(A)$

$\Rightarrow \underbrace{(\vec{b} - A\hat{x})}_{\in\, \text{null}(A^T)} \perp \text{col}(A)$

$\Rightarrow A^T(\vec{b} - A\hat{x}) = 0$

$\Rightarrow A^T A \hat{x} = A^T \vec{b}$ → normal equation

$\Rightarrow \hat{x} = (A^T A)^{-1} A^T \vec{b}$ → least squares solution

We consider $A\vec{x} = \vec{b}$ where A is $m \times n$ $(m \geq n)$ and the equation is possibly inconsistent

1) Replace eqn with the normal equations

$$(A^T A)\hat{x} = A^T \vec{b}$$

$\Rightarrow \hat{x} = (A^T A)^{-1} A^T \vec{b}$ gives the least square solution s.t. $\|A\vec{x} - \vec{b}\|$ is minimized

Ex- Regression Line;

We have a set of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ that we want to fit to a line $y = a + bx$

$\Rightarrow$ looking for

$$a + bx_1 = y_1$$
$$a + bx_2 = y_2$$
$$\vdots$$
$$a + bx_n = y_n$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{b}}$$

Normal Equation:

$$\underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{b}}$$

$$\begin{bmatrix} n & x_1 + x_2 + \dots + x_n \\ x_1 + x_2 + \dots + x_n & x_1^2 + \dots + x_n^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 + \dots + y_n \\ x_1 y_1 + \dots + x_n y_n \end{bmatrix}$$

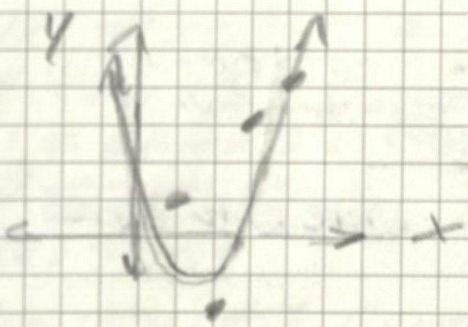
Ex 2. Let W be a subspace whose basis form columns of a matrix A

$$A = [\vec{v_1}\ \vec{v_2}\ \dots\ \vec{v_k}]$$

$$\text{proj}_W \vec{v} \Leftrightarrow A\vec{x} \Rightarrow A(A^TA)^{-1}A^T\vec{v}$$

$*$ if columns of A are orthonormal, $A^TA = I$

Ex 3 - Find the best fit parabola for points $(1,1), (2,-2), (3,3), (4,4)$



$$y = a + bx + cx^2$$
$$a + b + c = 1$$
$$a + 2b + 4c = -2$$
$$a + 3b + 9c = 3$$
$$a + 4b + 16c = 4$$

Normal Eqn:

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 22 \\ 34 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -18/5 \\ 1 \end{bmatrix} \rightsquigarrow y = 3 - \frac{18}{5}x + x^2$$

best fit parabola

Differential Equations:

Def. A differential equation is an equation that contains one or more unknown functions of one or more independent variables and their derivatives

Ex.

1) $y' = \sin x \xrightarrow{\text{sol}} y(x) = -\cos x + C$ ← all solutions

2) $\frac{dM}{dt} = 0.1 \cdot M \Rightarrow M(t) = M_0 e^{0.1t}$

order 1

Classification:

Ordinary DEs: one dep and ind vars

- $y'' - (3y')^2 + 17e^x y = \log x^3$ ← order 2

Partial DEs: > 1 dep and ind vars

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$$

Wave Eqn

Order: 1 dep var, 1 ind var

ODE: $F(x, y, y', y'', \dots y^{(n)}) = 0$

has order n

Linear: F linear eqn w/ respect to derivatives

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_0(x) y = f(x)$$

Non Linear:

$$y'' y''' - (y'^3 + e^y) = 0$$

Least-Squares with QR Factorization:

Note. $Q^T Q = I$

$$A = QR$$

$$A^T A\hat{x} = A^T \vec{b}$$

$$(QR)^T (QR)\hat{x} = (QR)^T \vec{b}$$

$$R^T Q^T Q R \hat{x} = R^T Q^T \vec{b}$$

$$R^T R \hat{x} = R^T Q^T \vec{b}$$

$$R\hat{x} = Q^T \vec{b}$$

Lecture Notes: 4.4.24

Ordinary DE: $y'' - y^2 y' + \sin x = 5$

Partial DE: $\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial y}{\partial x} \sin y \cdots$

Order = highest derivative point

Linear:

$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = f(x)$

ex. $e^x + y'' = y' - y = e^x$

Nonlinear:

ex. $y'' + (y')^2 = \sin x$ or $y' = e^y + x$

Autonomous and Nonautonomous DE:

Eqn does not explicitly contain the independent variable

$A \begin{cases} y'' - 2y' + e^y = 0 \\ \frac{dy}{dx} = \sin y \end{cases}$ $NA \begin{cases} y' = e^x + y \end{cases}$

Solution to DE:

Consider ODE $F(x, y, y', \dots y^{n}) = 0$

$y(x)$ is a solution to eqn on interval I if:

$\{F(x, y(x), y'(x), \dots, y^{n}(x)) = 0 ; \forall x \in I\}$

Ex. 1 → "particular solution"

$y'' - 2y' + y = 0$

$y(x) = xe^x$ is a solution on I $(-\infty, \infty)$

$y'' - 2y + y = 0$ $2e^x + xe^x - 2e^x - 2xe^x + xe^x = 0$

$y' = e^x + xe^x$

$y'' = 2e^x + xe^x$

$y(x) = xe^x$

Also: $y(x) = c_1 xe^x + c_2 e^x$

generalized solution

Ex. 2

$xy' + y = 0$ → 1st order NA ODE

C: $y(x) = \frac{1}{x}$ is a sol on $(-\infty, 0)$ and $(0, \infty)$

P: $y'(x) = -\frac{1}{x^2} \Rightarrow -\frac{x}{x^2} + \frac{1}{x} = 0$

Ex. 3

$y' + 2xy^2 = 0$

C: $y(x) = \dfrac{1}{x^2+C}$ is a solution

P: $y'(x) = \dfrac{-2x}{(x^2+C)^2} \Rightarrow \dfrac{-2x}{(x^2+C)^2} + 2x \cdot \dfrac{1}{(x^2+C)^2} = 0$

Requires $y(0) = -1$

$y(0) = \dfrac{1}{0^2+C} = -1 \Rightarrow C = -1$

$y(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)} \Rightarrow I: (-1, 1)$

Initial Value Problems (IVPs):

$F(x, y', \dots, y^n) = 0$

w/ conditions: $y(x_0) = y_0$ $\Rightarrow$ solution to DE that satisfies initial condition

$y'(x_0) = y_1$

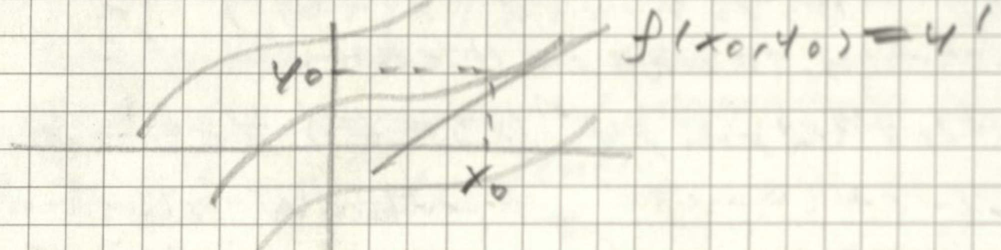
$\vdots$

$y^{(n-1)}(x_0) = y_n$

Geometrically,

1st order DE: $y' = f(x, y)$

Solutions occur in the plane



Autonomous First Order ODE's:

$y' = f(y)$

Can the constant fn: $y(x) = y_0$ be a solution?

$(y_0)' = f(y_0) \Rightarrow 0 = f(y_0)$

$\therefore \forall y_0$ s.t. $f(y_0) = 0$ gives constant sol $y(x) = y_0$

called an equilibrium solution

Ex. $y' = \sin y$, we have solutions

$y(t) = k\pi$ where $k = 0, \pm 1, \pm 2, \dots$

Ex. $y' = (y-1)(y+1)(y-2)$

$y(t) = 1$

$y(t) = -1$

$y(t) = 2$

Suppose we have $y(0) = \frac{3}{2}, \frac{15}{8}, \ldots$ s.t. $1 < y < 2$

$\lim_{t \to +\infty} y(t) = 1$

Rule. If $f(y)$ is between two equilibrium values $y_0 < y_1$, then any solution s.t. $y(t_0) \in (y_0, y_1)$ will have:

$$\lim_{t \to \infty} y(t) = y_1$$

If " ——— " ——— ": $\lim_{t \to +\infty} y(t) = y_0$

Lecture Notes: 4.10.24

Solutions to ODEs:

Ex- $y'' + 9y = 0$ linear, autonomous, 2nd order

IVP. $y(0) = 1$

$y'(0) = 2$

Verify that $y(x) = \underbrace{A\cos 3x + B\sin 3x}_{\text{general solution}}$ is a solution for any A, B

$y'(x) = -3A\sin 3x + 3B\cos 3x$

$y''(x) = -9A\cos 3x - 9B\sin 3$

$= -9(A\cos 3x + B\sin 3x)$

$\Rightarrow y''(x) + 9y(x) \equiv 0 \quad \forall x$

I.V.P: $y(0) = 1 \Rightarrow A \cdot 1 + B \cdot 0 = 1 \Rightarrow A = 1$

$y'(0) = 2 \Rightarrow -3 \cdot A \cdot 0 + 3 \cdot B \cdot 1 = 2 \Rightarrow B = \frac{2}{3}$

Direction Fields and Behavior of autonomous 1st order DE:

Ex: $y' = y^6 - 16y^2$

Equilibrium Solutions

$y(x) = C = y_0$

$\Rightarrow 0 = y_0^6 - 16y_0^2$

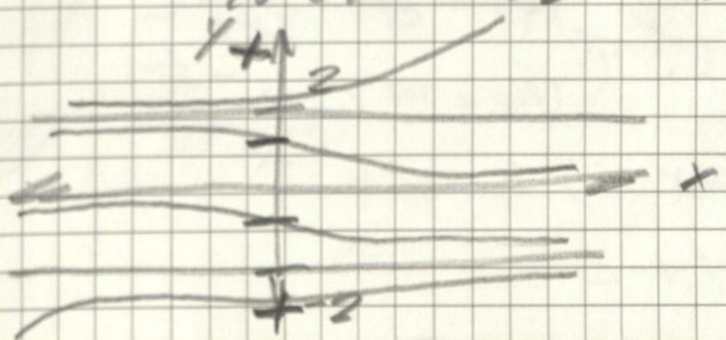
$= y_0^2(y_0^4 - 16) = y_0^2(y_0^2 - 4)(y_0^2 + 4)$

$y_0^2(y_0 - 2)(y_0 + 2)(y_0^2 + 4)$

$\Rightarrow y_0 = -2$

$y_0 = 0$

$y_0 = 2$



Seperable DE:

Def. A 1st order ODE is separable if it can be written as:

$$\frac{dy}{dx} = g(y)h(x)$$

antiderivative

Solution. $\frac{dy}{g(y)} = h(x)dx \Longleftrightarrow \underbrace{\int \frac{dy}{g(y)}}_{G(y)} = \underbrace{\int h(x)dx}_{H(x)}$

$\Rightarrow G(y) = H(x) + C$ ← general solution

IVPs: $y(x_0) = y_0$

$\Rightarrow G(y_0) = H(x_0) + C \Rightarrow C = G(y_0) - H(x_0)$

Ex. $y' = e^{x+y}$

$y' = e^x e^y$

$y' = x + y$ not separable

$\int \frac{dy}{e^y} = \int e^x dx \Rightarrow -e^{-y} = e^x + C$

$\Rightarrow -\frac{1}{e^y} = e^x + C \Rightarrow e^y = \frac{-1}{e^x + C} = \frac{1}{C - e^x}$

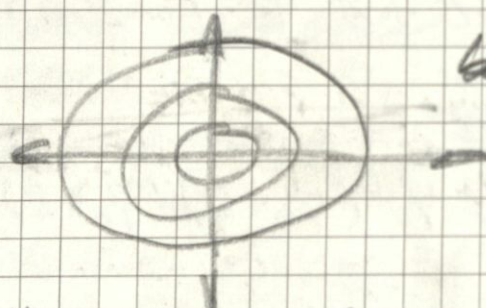
↑ different C since $C > 0$

$\Rightarrow y(x) = \ln\left(\frac{1}{C - e^x}\right)$

Ex. $y' = -\frac{x}{y}$

$\int y\,dy = \int -x\,dx \Rightarrow y^2 = -x^2 + C$

$\Longleftrightarrow x^2 + y^2 = R^2$ ← R since $C > 0$



better left as implicit solution (b/c it is a circle)

Ex. $y'' = \sin y \cos x$

not seperable (only 1st order are separable)

Ex. $x^2 \frac{dy}{dx} = y - xy$

$\Longleftrightarrow \frac{dy}{dx} = y\frac{(1-x)}{x^2}$

$\int \frac{dy}{y} = \int \left(\frac{1}{x^2} - \frac{1}{x}\right)dx$

$\ln|y| = -\frac{1}{x} - \ln|x| + c$

$|y| = e^{-\frac{1}{x}} \cdot \frac{1}{|x|} C$

$y(x) = C\frac{e^{-\frac{1}{x}}}{x}$

Linear 1st order ODEs:

$$a_1(x)y' + a_0(x)y = g(x)$$

known functions

$$\frac{a_0(x)}{a_1(x)} \qquad \frac{g(x)}{a_1(x)}$$

Standard Form: $y' + P(x)y = f(x)$

Solution: "Integrating Factor Trick"

Def. $\mu(x)$ as a solution to $\underbrace{\mu'(x) = P(x)\mu}_{\text{seperable}}$

$$\frac{d\mu}{\mu} = P(x)dx$$

$$\Rightarrow \ln\mu = \int P(x)dx \Rightarrow \mu(x) = e^{\int P(x)dx}$$

Consider:

$$(\mu(x)y)' = \mu'(x)y + \mu(x)y'$$

$$= P(x)\mu(x)y + \mu(x)(f(x) - P(x)y)$$

$$= \mu(x)f(x)$$

$$\Rightarrow (\underbrace{e^{\int P(x)dx}}_{\text{known}}y)' = \underbrace{e^{\int P(x)dx}f(x)}_{\text{known}}$$

$$\Rightarrow e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)dx + C$$

$$\Rightarrow y(x) = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}f(x)dx + C\right)$$

IVP. $y(x_0) = y_0$

$$y(x) = e^{-\int_{x_0}^{x} P(t)dt}\left(\int_{x_0}^{x} e^{\int_{x_0}^{s} P(t)dt} f(s)ds + y_0\right)$$

$$y(x_0) = y_0$$

Exam 3 Review: 4.11.24

4.1, 4.3: eigenvectors and eigenvalues

$A\vec{v} = \lambda\vec{v}$; λ - eigenvalue, $\vec{v}$ eigenvector

$\Rightarrow A\vec{v} - \lambda I\vec{v} = \vec{0} \Rightarrow (A-\lambda I)\vec{v} = \vec{0}$

(only way $A\vec{v} = \vec{0}$ $\forall \vec{v} \neq \vec{0}$ is if $\det A = 0$)

$\Rightarrow \det(A-\lambda I) = 0 \Rightarrow p(\lambda) = 0$ - characteristic eqn

(if all solutions are real)

$\Rightarrow (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\dots(\lambda-\lambda_k)^{n_k} = 0$

λ_1 has algebraic multiplicity n_1

λ_2 has algebraic multiplicity n_2

$\vdots$

λ_3 has algebraic multiplicity n_k

forms the null space

$\Rightarrow A\vec{v} = \lambda_i\vec{v}$ ($\vec{v} \neq 0$) $\Rightarrow (A-\lambda_i)\vec{v} = 0$

$\Rightarrow E_{\lambda_i} = \{\vec{v} : A\vec{v} = \lambda_i\vec{v}\}$ - eigenspace of λ_i

$\Rightarrow \dim E_{\lambda_i}$ - geometric multiplicity of λ_i

$\Rightarrow P = [\vec{v}_{\lambda_1,1} \dots \vec{v}_{\lambda_1,n_1}\ \vec{v}_{\lambda_2,1} \dots \vec{v}_{\lambda_2,n_2} \dots \vec{v}_{\lambda_k,1} \dots \vec{v}_{\lambda_k,n_k}]$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 \ddots \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_k & \\ 0 & & & & & & \ddots \ \lambda_k \end{bmatrix}$$

$$\underbrace{\qquad}_{n_1} \quad \underbrace{\qquad}_{n_2} \quad \dots \quad \underbrace{\qquad}_{n_k}$$

Problems:

$\lambda_i \in \mathbb{C} \Rightarrow A$ not diagonalizable (describes rotation)

$\dim E_{\lambda_i} \neq n_i \Rightarrow A$ not diagonalizable

(not enough eigenvectors to span the eigenspace)

Properties:

1) A not invertable $\iff$ 0 is an eigenvalue
2) λ is an eigenvalue of $A \iff \lambda^k$ is an eigenvalue of A^k
3) A is invertable and $A\vec{v} = \lambda\vec{v} \iff A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$

4.4: Similarity

$A \sim B \iff B = P^{-1}AP$ $\forall$ invertable P

B is diagonal $\iff A$ is diagonalizable

Properties:

1) $A \sim B$, $B \sim C \Rightarrow A \sim C$
2) $A \sim B$, A is diagonalizable $\Rightarrow B$ is diagonalizable
3) $A \sim B \Rightarrow \det(A-\lambda I) = \det(B-\lambda I)$
   $\Rightarrow$ same eigenvalues and multiplicities

4) $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $A \Rightarrow \det A = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n$
5) $A \sim B \Leftrightarrow B = P^{-1}AP \Rightarrow B^k = P^{-1}A^kP \ \forall k = 1, 2, \ldots$
if A also invertable, then $\forall k$

4.4: diagonalization $\Rightarrow A \sim B \Leftrightarrow B = P^{-1}AP$
B is diagonal $\Leftrightarrow$ A is diagonalizable
$B = P_{\mathcal{B} \leftarrow \mathcal{E}} [T]_{\mathcal{E} \leftarrow \mathcal{E}} P_{\mathcal{E} \leftarrow \mathcal{B}}$

4.6: complex eigenvalues
$n = 2$, no real eigenvalues
$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ — common form. ex. $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ rotation matrix
Define. $r = \sqrt{a^2 + b^2}$
1) $r > 1 \Rightarrow A\vec{x}, A^2\vec{x}, \ldots \Rightarrow$ spiral out
2) $r < 1 \Rightarrow A\vec{x}, A^2\vec{x}, \ldots \Rightarrow$ spiral in
3) $r = 1 \Rightarrow A\vec{x}, A^2\vec{x}, \ldots \Rightarrow$ ellipse (circle)
diagonalization $\Leftrightarrow$ basis is eigenvectors
similarity $\Leftrightarrow$ basis is real and imaginary
Suppose:
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has $\lambda_{1,2} = a \pm bi$
$\vec{v} = \mathrm{Re}\,\vec{v} + i\,\mathrm{Im}\,\vec{v} \Rightarrow P = [\mathrm{Re}\,\vec{v} \ \mathrm{Im}\,\vec{v}]$
$\Rightarrow P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

1.2, 5.1, 5.2: orthogonality, orthogonal complements
Inner product: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\alpha = u_1v_1 + u_2v_2$
Also, $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + \ldots u_n^2}$ and $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$
Properties:
1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $(\vec{u} \cdot \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
3) $c\vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4) $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = 0$
$\vec{u} \cdot \vec{v} = [u]^T \vec{v}$
$\mathrm{proj}_{\vec{u}} \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

Orthogonal System: $\{\vec{v}_1, \ldots, \vec{v}_k : \vec{v}_i \cdot \vec{v}_j = 0 \ \forall i \neq \forall j\}$
Property. $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent
Orthogonal Basis: $\{\vec{v}_1, \ldots, \vec{v}_n : \vec{v}_i \cdot \vec{v}_j = 0 \ \forall i \neq \forall j\} \in \mathbb{R}^n$
→ same # of vectors as dimension of space
Orthonormal Basis:
$Q = \{\vec{v}_1, \ldots \vec{v}_n : \vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}\} \in \mathbb{R}^n$
* preserves length, dot product, and orthogonality
$\Rightarrow Q^TQ = I \Rightarrow Q^T = Q \Rightarrow Q = Q^{-1}$

Orthogonal Complements:

$\vec{v} \perp W \Rightarrow \vec{v} \in W^{\perp} = \{ \forall \vec{v} \in \mathbb{R}^n : \vec{v} \perp W \in \mathbb{R}^n \}$

$(\text{row}(A))^{\perp} = \text{null}(A)$

$(\text{col}(A))^{\perp} = \text{null}(A^T)$

Properties

1) $(W^{\perp})^{\perp} = W$
2) $W^{\perp} \subset \mathbb{R}^n$
3) $\dim W + \dim W^{\perp} = \dim \mathbb{R}^n \ \forall W \subset \mathbb{R}^n$

5.1, 5.2: orthogonal projection, orthonormal sets

$\vec{v} = \text{proj}_{\vec{u}} \vec{v} + \text{perp}_{\vec{u}} \vec{v} \Rightarrow \text{perp}_{\vec{u}} \vec{v} = \vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$

$\text{dist}(\vec{v} \cdot W) = |\text{perp}_W \vec{v}|$

$W = \text{span}\{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k \} \in \mathbb{R}^n$

$\text{proj}_W \vec{v} = \text{proj}_{\vec{u}_1} \vec{v} + \text{proj}_{\vec{u}_2} \vec{v} + \ldots + \text{proj}_{\vec{u}_k} \vec{v}$

if $\vec{v} \in W \Leftrightarrow \text{proj}_W \vec{v} = \vec{v}$ and $\text{perp}_W \vec{v} = 0$

$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \text{proj}_{\vec{u}_1} \vec{v} \\ \vdots \\ \text{proj}_{\vec{u}_n} \vec{v} \end{bmatrix} \ \forall \text{OB } \mathcal{B} \in \mathbb{R}^n$

5.3: Gram-Schmidt Process

$\forall \mathcal{B} \in W = \{ \vec{x}_1, \vec{x}_2, \ldots \vec{x}_k \}$ produce OB $\{ \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \}$

1) $\vec{v}_1 = \vec{x}_1$, $W_1 = \text{span}\{ \vec{v}_1 \}$
2) $\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$, $W_2 = \text{span}\{ \vec{v}_1, \vec{v}_2 \}$

$\vdots$

k) $\vec{v}_k = \vec{x}_k - \text{proj}_{W_{k-1}} \vec{x}_k$, $W_k = \text{span}\{ \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \}$

*need to normalize to make ONB

QR Factorization:

$A = [\vec{x}_1 \ldots \vec{x}_k]$, $Q = [\vec{q}_1 \ldots \vec{q}_k]$ → orthonormal columns

→ upper triangular matrix

$A = [\vec{x}_1 \ldots \vec{x}_k] = \underbrace{[\vec{q}_1 \ldots \vec{q}_k]}_{Q} \underbrace{\begin{bmatrix} r_{11} & \cdots & r_{1k} \\ & \ddots & \vdots \\ 0 & & r_{kk} \end{bmatrix}}_{R}$

$\Rightarrow R = Q^{-1} A = Q^T A$

7.3: Least Squares Solutions

Consider inconsistent $A\vec{x} = \vec{b}$.

Find $A\hat{x} = \vec{b}$ s.t. $|A\hat{x} - \vec{b}|$ is minimized

$A\hat{x} = \text{proj}_{\text{col}(A)} \vec{b}$

$\rightsquigarrow A^T A \hat{x} = A^T \vec{b}$ → normal equations

$\Rightarrow \hat{x} = (A^T A)^{-1} A^T \vec{b}$ → least squares solution

Lecture Notes: 4.12.24

1st order linear ODE: (*)

$\frac{dy}{dx} + P(x)y = f(x)$, $\mu(x) = e^{\int P(x)dx}$ integrating factor

$y = \frac{1}{\mu(x)}\left(\int \mu(x) f(x)\, dx + C\right)$ (*) standard form

Ex. $x\frac{dy}{dx} + 3y = x^{-2}e^{x}$, $x > 0$

$\Rightarrow \frac{dy}{dx} + \frac{3}{x}y = x^{-3}e^{x}$, $\mu(x) = e^{\int \frac{3}{x}dx} = x^{3}$

$\Rightarrow \frac{d}{dx}(x^3 y) = x^3 \cdot x^{-3}e^{x}$ general solution

$\Rightarrow y = \frac{1}{x^3}\int x^3 \cdot x^{-3}e^{x}dx + C = \frac{1}{x^3}(e^{x} + C)$

Ex. IVP

$\begin{cases} x\frac{dy}{dx} + 3y = x^{-2}e^{x} \rightsquigarrow y = \frac{1}{x^3}(e^{x}+C) \\ y(1) = 0 \longrightarrow 0 = (e^{1} + C) \Rightarrow C = -e \end{cases}$

$\Rightarrow y = \frac{1}{x^3}(e^{x} - e)$

Exact Equations

1st order ODE in differential form:

$M(x,y)dx + N(x,y)dy = 0$ (*)

is exact if there is some function $f(x,y)$ so:

$\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$

In that case,

(*) $\Leftrightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

"increment of f as we change from $(x,y) \to (x+dx, y+dy)$"

$\Leftrightarrow f(x,y) = C$ (implicit solution of (*))

Sometimes this is impossible, so we have a test:

If (*) is exact,

$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ works with other way as well

"constant" of integration

How to find f:

1) $\frac{\partial f}{\partial x} = M(x,y) \Rightarrow f(x,y) = \int M(x,y)dx + g(y)$

2) Plug calculated $f(x,y)$ into $\frac{\partial f}{\partial y} = N$, calculate $g'(y)$, integrate to get $g(y)$ and therefore f

Ex. $2xy\,dx + (x^2-1)\,dy = 0$ (*)

$\frac{\partial M}{\partial y} = 2x \overset{\checkmark}{=} \frac{\partial N}{\partial x} = 2x$ ∴ exact

$\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x,y) = \int 2xy\,dx + g(y) = x^2y + g(y)$

$\frac{\partial f}{\partial y} = x^2 - 1 = x^2 + g'(y) \Rightarrow g'(y) = -1$

$\Rightarrow g(y) = \int g'(y)\,dy = -y \Rightarrow f(x,y) = x^2y - y$

Solution of (*) is $x^2y - y = C$

Integrating Factors:

If $M(x,y)\,dx + N(x,y)\,dy = 0$ (*) is not exact, we can make it exact by multiplying by an integrating factor: $\mu(x,y)$

How to find μ? (cannot always)

$\mu M\,dx + \mu N\,dy = 0$ (*)

$(\mu M)_y \overset{?}{=} (\mu N)_x \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$

$\Rightarrow \mu_x N - \mu_y M = \mu(M_y - N_x)$ (#) PDE ∴ hard

Case $\mu = \mu(x)$:

$\Rightarrow \mu_x N = \mu(M_y - N_x) \Rightarrow \frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$

(if this is a function of x alone, solve 1st order separable eqn)

Case $\mu = \mu(y)$:

$\frac{\mu_y}{\mu} = -\frac{M_y - N_x}{M}$ (if this depends on y alone, can solve to get IF $\mu(y)$)

Ex. $\underbrace{xy}_{M}\,dx + \underbrace{(2x^2+3y^2)}_{N}\,dy = 0$

$M_y = x \neq N_x = 4x$ ∴ not exact

look for IF:

$\frac{M_y - N_x}{N} = \frac{x-4x}{2x^2+3y^2} = -\frac{3x}{2x^2+3y^2}$ depends on y ∴ no IF $\mu(x)$

$-\frac{M_y - N_x}{M} = -\frac{x-4x}{xy} = \frac{3x}{xy} = \frac{3}{y}$ depends only on y ∴ ∃ IF $\mu(y)$

$\Rightarrow \frac{\mu_y}{\mu} = \frac{1}{\mu}\frac{d\mu}{dy} = \frac{3}{y} \Rightarrow \int \frac{1}{\mu}\,d\mu = \int \frac{3}{y}\,dy$

$\Rightarrow \ln \mu = 3\ln y \Rightarrow \mu = y^3$

$(*)\,y^3 \Rightarrow \tilde{M}\,dx + \tilde{N}\,dy = 0$ s.t. $\tilde{M}_y \overset{\checkmark}{=} \tilde{N}_x$

Lecture Notes:

Modeling w/ DE:

Exs -

1) Growth / Decay Models

a) rate of growth for investment is $k\%$ annually compounded continuously

$$\frac{dM}{dt} = kM \quad , \quad M(0) = M_0$$

in t years, you will get

$$\Rightarrow M(t) = M_0 e^{kt}$$

b) radioactive decay

$$\frac{da}{dt} = -ka \Rightarrow a(t) = a_0 e^{-kt}$$

2) Newton's Law of cooling: the rate of change of temperature of an object is proportional to the difference between its current temp and the temp of surrounding media

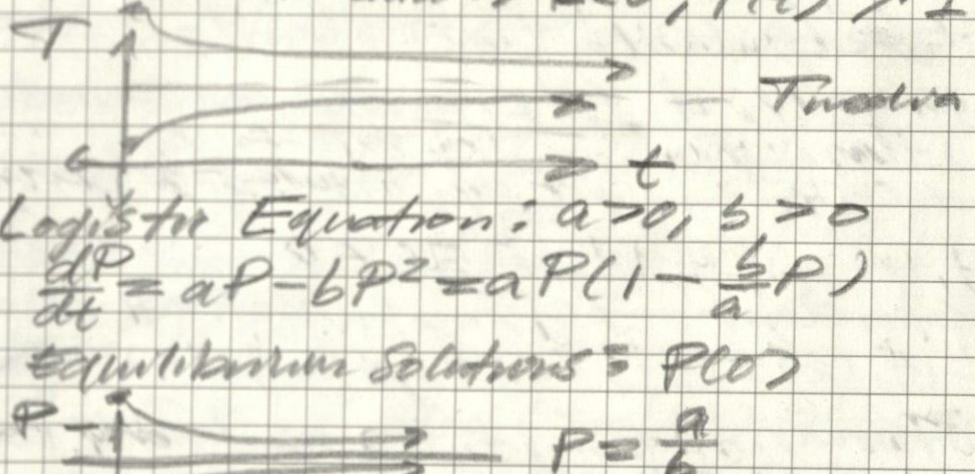
$$\frac{dT}{dt} = k(T - T_{media}) \; , \quad T(0) = T_0$$

unknown? $\leftarrow$ constant

$$\Rightarrow T(t) = T_{media} + (T_0 - T_{media})e^{kt}$$

if $T_0 > T_{media} \Rightarrow k < 0, \; T(t) \searrow T_{media}$

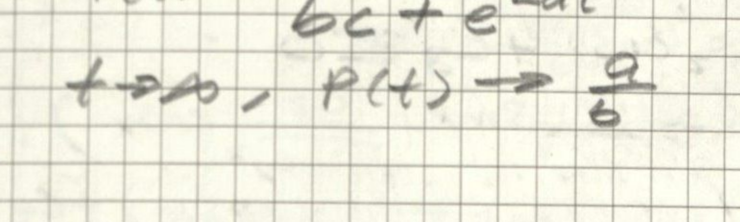
if $T_0 < T_{media} \Rightarrow k < 0, \; T(t) \nearrow T_{media}$



3) Logistic Equation: $a > 0, \; b > 0$

$$\frac{dP}{dt} = aP - bP^2 = aP\left(1 - \frac{b}{a}P\right)$$

Equilibrium Solutions: $P(0)$



$$\Rightarrow P(t) = \frac{ac}{bc + e^{-at}}$$

$$t \to \infty, \; P(t) \to \frac{a}{b}$$

Linear DEs:

$x \Rightarrow$ independent variable

$y = y(x) \rightarrow$ dependent variable (unknown)

Solve: $a_n(x) + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = g(x)$

order n, $a_0(x), \dots, a_n(x), g(x)$ are known functions

Find solutions s.t.:

$y(x_0) = y_0,\ y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$

Thm: Existence and Uniqueness

Suppose $a_0(x), \dots, a_n(x), g(x)$ are continuous on an interval I and $a_n(x) \neq$ on I.

Then IVP above has a unique solution

If $g(x) \equiv 0$ on an interval I then our linear equation is homogeneous ($\Leftrightarrow$ identically equal to)

Key Feature: superposition principle

If $y_1(x)$ and $y_2(x)$ are two solutions to

$a_n(x) y^n + \dots + a_0(x) y = 0 \quad (*)$

Then $c_1 y_1(x) + c_2 y_2(x)$ is also a solution to $*$ for and c_1, c_2

Why? $a_n(x)(c_1 y_1^{(n)} + c_2 y_2^{(n)}) + \dots + a_0(x)(c_1 y + c_2 y_2)$

$= c_1 (a_n(x) y_1^n + \dots + a_0(x) y_1) = 0$

$+ c_2 (a_n(x) y_2^n + \dots + a_0(x) y_2) = 0 \Rightarrow 0$

More generally, if $y_1 y_2, \dots, y_k$ are solutions to $(*)$ then $\forall c_1, \dots, c_k$

$c_1 y_1 + c_2 y_2 + \dots + c_k y_k$ is also a sol to $(*)$

Def. $f_1(x), \dots, f_n(x) \in I \in \mathbb{R}$ are linearly dependent $\exists c_1, \dots, c_n$ not all zero st.

$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \ \forall x \in I$

otherwise $f_1(x), \dots, f_n(x)$ are linearly independent

Ex. 1) $\sin^2 x, \cos^2 x, 1 \in \mathbb{R}^n$ $\quad \rightarrow = 0 \ \forall x$

$1 \cdot \sin^2 x + 1 \cdot \cos^2 x - 1 \cdot 1 \equiv 0$

2) $\sin x, \cos x, 1$

$c_1 \sin x + c_2 \cos x + c_3 \equiv 0$

$\Rightarrow \sqrt{c_1^2 + c_2^2} \left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \sin x + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \cos x \right) + c_3 = 0$

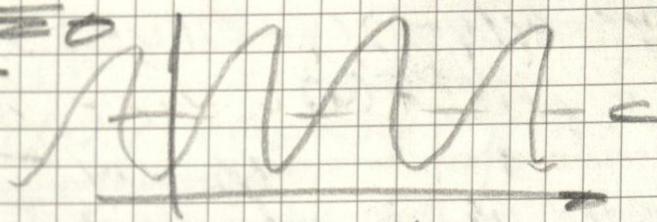
$\Rightarrow \sqrt{c_1^2 + c_2^2} (\cos\alpha \sin x + \sin\alpha \cos x) + c_3 = 0$

$\Rightarrow A\sin(x+\alpha)+c_3 \equiv 0$

$\therefore$ linearly independent

In fact, {sin x, cos x, 1} are solutions to:

$y''' + y' = 0$



→ examples of a fundamental system of solutions to a homogeneous linear DE

<u>How to check lin. independence?</u>

Suppose $c_1 f_1(x) + \dots + c_n f_n(x) \equiv 0$

$\forall$ smooth $f_i(x) \rightarrow$ infinitely differentiable

$$\Rightarrow \begin{cases} c_1 f_1'(x) + \dots + c_n f_n'(x) = 0 \\ \vdots \\ c_1 f^{(n-1)}(x) + \dots + c_n f_n^{(n-1)}(x) = 0 \end{cases}$$

n equations for n constants $c_1, \dots, c_n$

$$\Rightarrow \underbrace{\begin{bmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{bmatrix}}_{A(f_1,\dots,f_n)} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow \det A(f_1, \dots, f_n) = 0 \Leftrightarrow f_1, \dots, f_n$ lin independent

<u>Lecture Notes</u>: Exam 3 Review 4.17.24

3. $L = \text{span}\left\{ \vec{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \right\}$. $\vec{u}\cdot\vec{u} = 1$

$$\text{proj}_L(\vec{x}) = \frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \left(\frac{3}{5}x_1 - \frac{4}{5}x_2\right)\begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} = \begin{bmatrix} \frac{9}{25}x_1 - \frac{12}{25}x_2 \\ -\frac{12}{25}x_1 + \frac{16}{25}x_2 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\text{proj}_L \vec{x} = QQ^T\vec{x}$ when $L = \text{col}(Q)$ and Q is orthonormal

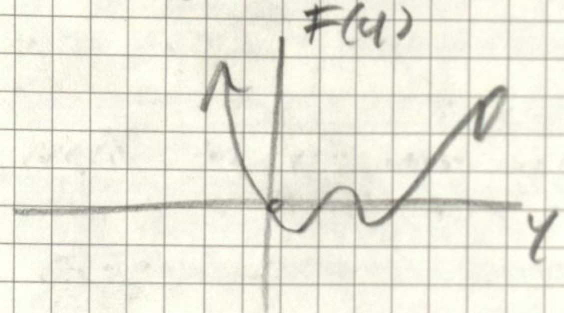
$\text{col}\, Q = \{Q\vec{y}\}$, $\Leftrightarrow \text{proj}_L(Q\vec{y}) = QQ^TQ\vec{y} = Q\vec{y}$

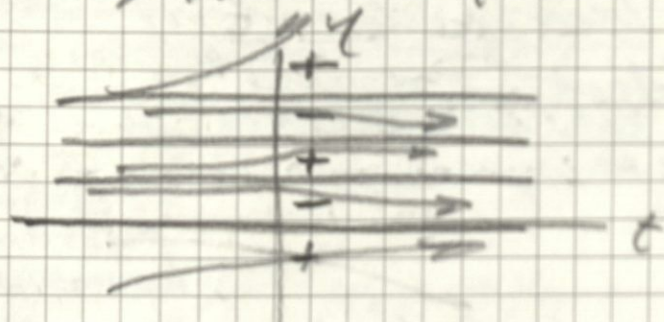
* $\dim \text{Ker}(A - \lambda_k I) \leq n^k$

→ CAN be less than

but CAN'T be diagonalizable in this case

6. $y' = y(y-1)(y-2)(y-3)$; $y(0) = 2.99$





$\lim_{t \to +\infty} y(t) = 2$ since $y(0) = 2.99$ and that portion decreases until next equilibrium solutions

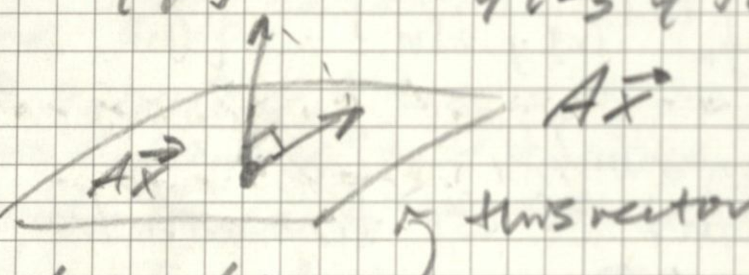
10. $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$. $A\hat{x} = \vec{b} \Rightarrow \hat{x} = ?$

$\therefore \text{proj}_{col(A)} \hat{x} \approx \vec{b} \Leftrightarrow A^T A \hat{x} = A^T \vec{b}$

$$\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 0 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = \frac{1}{9} \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

b)



4. a) always true
b) $A^T A$ always becomes invertable since columns are lin ind, so $\det(A^T A) \neq 0$
<u>Note</u>. $\det(AA^T) = 0$ so e

1. $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda i \\ \lambda 1 \end{bmatrix} \Rightarrow 1 \cdot i + 1 = \lambda$

5. $A_{2\times3}$. $AA^T = 3\times3$
$A^T A = 2\times2$

2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}$
$\dim W = 2 \therefore \dim W^\perp = 1$ since $W + W^\perp = \dim \mathbb{R}^3$

* to use $\text{proj}_W \vec{v}$ formula W has to have an orthogonal basis

Lecture Notes: 4.22.24

Lecture Notes:

2nd Order Linear ODE:

Standard Form: $y'' + a_1(x)y' + a_0(x)y = b(x)$ $(*)$

- if $b(x) = 0$, $(*)$ is homogeneous
- $\forall$ homogeneous $(*)$, $\exists$ fundamental set of solutions $y_1, y_2 \Leftrightarrow W(y_1, y_2)(x) = y_1(x) \cdot y_2'(x) - y_1'(x)y_2(x) \neq 0$
- $W(y_1, y_2)(x) = Ce^{-\int a_1(x)dx}$
  $\Rightarrow W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} a_1(s)ds}$
- $\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2}$ Can be used to find y_2, if y_1 is known

Ex: $xy'' - (2x+1)y' + (x+1)y = 0$

- Find $y_2(x)$ if it is known that $y_1(x) = e^x$ is a solution

1) Rewrite eqn in standard form

$$y'' \underbrace{-\left(\frac{2x+1}{x}\right)}_{a_1(x)} y' + \underbrace{\left(\frac{x+1}{x}\right)}_{a_0(x)} y = 0$$

2) Find an antiderivative of $a_1(x)$, $x > 0$

$$\int \frac{2x+1}{x} dx = -\int \left(2 + \frac{1}{x}\right)dx = -2x - \ln x$$

3) Find a Wronskian:

$$W(y_1, y_2) = e^{2x + \ln x} = xe^{2x}$$

4) $\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2} = \frac{xe^{2x}}{e^{2x}} = x$

$$\Rightarrow \left(\frac{y_2}{y_1}\right) = \frac{x^2}{2} \Rightarrow y_2(x) = \frac{x^2}{2} y_1(x) = \frac{x^2}{2} e^x$$

$\Rightarrow$ Conclusion: e^x, x^2e^x form a fundamental system of solutions.

A general solution to ODE is: $C_1e^x + C_2x^2e^x$

Wronskian:

$$W(f_1, \ldots, f_n)(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

Ex. nonhomogeneous

$xy'' - (2x+1)y' + (x+1) = x^2$ has $\phi(x) = x+1$ as a solution. Solve IVP st. $y(1) = 2$, $y'(1) = 3$

Note. $\phi(x)$ + any solution to homogeneous eqn is still a solution to eqn above

$y(x) = \phi(x) + c_1 e^x + c_2 x^2 e^x$

To satisfy IVP:

$2 = y(1) = \phi(1) + c_1 e + c_2 e$

$3 = y'(1) = \left(1 + c_1 e^x + c_2(2xe^x + x^2 e^x)\right)\Big|_{x=1}$

$= 1 + c_1 e + c_2 \cdot 3e$

$\Rightarrow \begin{matrix} c_1 e + c_2 e = 0 \\ c_1 e + 3c_2 e = 2 \end{matrix} \Rightarrow c_2 = \frac{1}{e},\ c_1 = -\frac{1}{e}$

$\Rightarrow$ Solution: $y(x) = x + 1 - e^{x-1} + x^2 e^{x-1}$

Linear, 2nd order ODE w/ constant coefficients:

$y'' + ay' + by = g(x)$

I. Homogeneous case: $g(x) = 0$

Hint. (1st order case)

$y' + ay = 0 \Rightarrow y' = -ay$

$\Rightarrow y(x) = Ce^{-ax}$

Try the same form of a solution for 2nd order eqn

let $y(x) = e^{\lambda x}$, λ - constant

$\Rightarrow y'(x) = \lambda e^{\lambda x}$

$\Rightarrow y''(x) = \lambda^2 e^{\lambda x}$

Plug in: $y'' + ay' + by = \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x}$

$\Rightarrow (\lambda^2 + a\lambda + b) e^{\lambda x} \stackrel{?}{=} 0 \quad \forall x$

$\Rightarrow e^{\lambda x}$ is a solution to ODE

$\Leftrightarrow \lambda^2 + a\lambda + b = 0 \Rightarrow$ find zero's λ_1, λ_2

and then $e^{\lambda_1 x}, e^{\lambda_2 x}$ will be fundamental system of solutions

General Solutions:

1) Solve characteristic eqn $\lambda^2 + a\lambda + b = 0$

Then: 1) $\lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

(?) 2) $\lambda_1 = \lambda_2 \Rightarrow y_1(x) = e^{\lambda_1 x}$, $y_2(x) = x e^{\lambda_1 x}$

How to get second solution?

Suppose $\lambda_2 = \lambda_1 + \varepsilon$

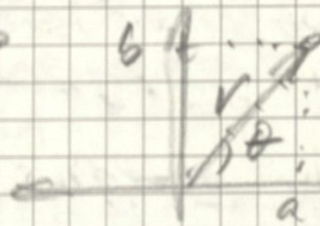
$$\frac{e^{(\lambda_1+\varepsilon)x} - e^{\lambda_1 x}}{\varepsilon} \xrightarrow[\varepsilon \to 0]{\text{L'Hôp}} \frac{x e^{\lambda_1 x} - 0}{1}$$

$\Rightarrow y(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

3) $\lambda_{1,2} = \alpha \pm i\beta$

General Sol: $c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = y(x)$

Recall. $a + ib = re^{i\theta}$



Euler's Formula

In Particular,

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{i\pi} + 1 = 0$

Euler's:

$\Rightarrow \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$, $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$\Rightarrow y(x) = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$\Rightarrow y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Ex. $y'' - 3y' + 2y = 0$ $\quad y(x) = c_1 e^{x} + c_2 e^{2x}$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

Ex. $y'' + 6y' + 9y = 0$ $\quad y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

$\lambda^2 + 6\lambda + 9 = 0$

$\Rightarrow (x+3)^2 = 0$

Ex. $y'' + 16y = 0$ $\quad y(x) = A\cos 4x + B\sin 4x$

$\lambda^2 + 16 = 0$

$\Rightarrow \lambda = \pm 4i$

Lecture Notes: Characteristic Eqn: 4.24.24

$y'' + ay' + by = 0 \Rightarrow r^2 + ar + b = 0$

Roots r_1, r_2: General Solution:

$r_1 \neq r_2 \in \mathbb{R} \Rightarrow c_1 e^{r_1 x} + c_2 e^{r_2 x}$

$r_1 = r_2 \in \mathbb{R} \Rightarrow c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

$r_{1,2} = \alpha \pm i\beta \in \mathbb{C} \Rightarrow e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Nonhomogeneous Eqns: Solution:

$y'' + ay' + by = g_1(x)$ $\quad y_1(x)$

$y'' + ay' + by = g_2(x)$ $\quad y_2(x)$

$\vdots$ $\quad \vdots$

$y'' + ay' + by = g_k(x)$ $\quad y_k(x)$

then, $y(x) = y_1(x) + y_2(x) + \dots + y_k(x)$ is a solution to $y'' + ay' + by = g_1(x) + g_2(x) + \dots + g_k(x)$

In Particular, if

y_p is a fixed solution to $y'' + ay' + b = g(x)$, then the general solution is:

$c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

fundamental set of solutions for homogeneous eqn

"Special" $g(x) \rightsquigarrow$ Method of undetermined coefficients

$\Leftrightarrow$ means $g(x)$ is made of polynomials, exponentials, sines, and cosines

Consider. $y'' - 3y' + 2y = g(x)$ of several types

Char. Eqn: $r^2 - 3r + 2 = 0 \Longrightarrow$ Gen Solution for
$(r-2)(r-1) = 0$ homog. eqn is
$c_1 e^x + c_2 e^{2x}$

1) $g(x) = e^{-x}$

Try $y_p(x) = Ae^{-x}$, $y_p' = -Ae^{-x}$, $y_p''(x) = Ae^{-x}$

$y_p'' - 3y_p' + 2y_p \stackrel{?}{=} e^{-x}$

$e^{-x}(A + 3A + 2A) \stackrel{?}{=} e^{-x} \Rightarrow 6A = 1 \Rightarrow A = \frac{1}{6}$

$\Rightarrow y_p(x) = \frac{1}{6} e^{-x}$

2) $g(x) = x^2+1$ $\Rightarrow$ $y_p'(x) = 2Ax + B$

Try $y_p(x) = Ax^2 + Bx + C$ $\quad y_p''(x) = 2A$

$\Rightarrow 2A - 3(2Ax+B) + 2(Ax^2+Bx+C) \overset{?}{=} x^2+1$

$\Rightarrow 2A = 1 \qquad \Rightarrow A = \frac{1}{2}$

$-6A + 2B = 0 \qquad B = \frac{3}{2}, \quad C = \frac{9}{4}$

$2A - 3B + 2C = 1$

3) $g(x) = \sin 2x$

Try $y_p(x) = A\cos 2x + B\sin 2x$

$y_p'(x) = -2A\sin 2x + 2B\cos 2x$

$y_p''(x) = -4A\cos 2x - 4B\sin 2x$

$\Rightarrow -4(A\cos 2x + B\sin 2x) + 6(-A\sin 2x + B\cos 2x)$

$+ 2(A\cos 2x + B\sin 2x) \overset{?}{=} \sin 2x$

$\Rightarrow -4A - 6B + 2A = 0 \rightarrow$ coeff of $\cos 2x$

$-4B + 6A + 2B = 1 \rightarrow$ coeff of $\sin 2x$

$\Rightarrow 2A = -6B \Rightarrow A = -3B$

$-4B - 18B + 2B = 1 \Rightarrow B = -\frac{1}{20}, \ A = \frac{3}{20}$

$\Rightarrow y_p(x) = \frac{3}{20}\cos 2x - \frac{1}{20}\sin 2x$

4) $g(x) = e^x$

$y_p(x) = Ae^x$ cannot work b/c already a solution to homog eqn

Instead, $y_p(x) = Axe^x$

$\Rightarrow y_p'(x) = A(e^x + xe^x)$

$y_p''(x) = A(2e^x + xe^x)$

$\Rightarrow A(2e^x + xe^x) - 3A(e^x + xe^x) + 2Axe^x \overset{?}{=} e^x$

$\Rightarrow e^x(2A - 3A) + xe^x(A - 3A + 2A) \overset{?}{=} e^x$

$\Rightarrow -Ae^x = e^x \Rightarrow A = -1$

$\Rightarrow y_p(x) = -xe^x$

* General Solution to nonhomogeneous solution is general solution to homogeneous solution plus the particular solution

For $y'' + ay' + by = g(x) \rightsquigarrow \lambda^2 + a\lambda + b = 0$

$g(x)$: $\qquad$ $y_p(x)$ =

$P(x) = p_n x^n + \dots + p_1 x + p_0 \Rightarrow x^s (A_n x^n + \dots + A_1 x + A_0)$

multiplicity $\rightsquigarrow$ s = # of times 0 is a solution to char. eqn.

$(p_n x^n + \dots + p_0) e^{ax} \Rightarrow x^s (A_n x^n + \dots + A_0)$ $\quad$ s = # of times a is a sol.

$(p_n x^n + \dots + p_0) \sin \beta x \cdot e^{\alpha x} + (Q_n x^n + \dots + Q_0) \cos \beta x \cdot e^{\alpha x} \Rightarrow x^s e^{\alpha x} ((A_n x^n + \dots + A_0) \cos \beta x + (B_n x^n + \dots + B_0) \sin \beta x)$

s = " — ″ — " $\alpha + i\beta$

General rule for nonhomogenous eqn

What if $g(x) = \underbrace{x^2 \cos 3x}_{g_1(x)} + \underbrace{e^{-2x}}_{g_2(x)} + \underbrace{(x^3 - 3)}_{g_3(x)}$

$y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x) = y_p(x)$

Ex: $y'' - 2y' + 2 = x e^x \cos x$ $\qquad$ $y_p(x) = x \begin{pmatrix} (A_1 x + A_0) \cos x \\ + (B_1 x + B_0) \sin x \end{pmatrix}$

$\lambda^2 - 2\lambda + 2 = 0$

$(\lambda - 1)^2 + 1 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i$ $\qquad$ $s = 1$

General Sol: $y(x) = y_p(x) + e^x (c_1 \cos x + c_2 \sin x)$

Lecture Notes 3 $\qquad$ 4.26.24

Undetermined coefficient Method for 2nd order linear ODEs with specific $g(x)$:
polynomial, polynomial · exponential, polynomial · sin/cos

Variation of Parameter Method works for any $g(x)$:

$y'' + a_1(x) y' + a_0(x) y = g(x)$ $\qquad$ (*)

Let $y_1(x), y_2(x)$ are a fundamental system of solutions for homogeneous eqn:

$y_1'' + a_1(x) y_1' + a_0(x) y_1 = 0$

$y_2'' + a_1(x) y_2' + a_0(x) y_2 = 0$

look for particular solution to (*):

$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$

$y_p = u_1 y_1 + u_2 y_2$

$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{= 0 \text{ "requirement" (condition)}} + (u_1 y_1' + u_2 y_2')$

$y_p'' = u_1 y_1'' + u_2 y_2'' + (u_1' y_1' + u_2' y_2')$

$y_p'' + a_1(x) y_p' + a_0(x) y_p \overset{?}{=} g(x)$

$\overset{?}{=} (u_1' y_1' + u_2' y_2') + u_1 (y_1'' + a_1(x) y_1' + a_0(x) y_1)$
$+ u_2 (y_2'' + a_1(x) y_2' + a_0(x) y_2) \overset{?}{=} g(x)$

$\Rightarrow u_1' y_1' + u_2' y_2' = g(x)$
(since remaining parameters $= 0$ due to solution y_1, y_2 to the homogeneous eqn given)

$\Rightarrow \begin{array}{l} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = g(x) \end{array} \Leftrightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}$

$\Rightarrow u_1' = \dfrac{-g(x) y_2}{W(y_1, y_2)}, \quad u_2' = \dfrac{g(x) y_1}{W(y_1, y_2)}$

Cramers Rule

As a result:

$$y_p(x) = y_1(x) \int_{x_0}^{x} \frac{-g(s) y_2(s)}{W(y_1, y_2)(s)} ds + y_2(x) \int_{x_0}^{x} \frac{g(s) y_1(s)}{W(y_1, y_2)(s)} ds$$

Ex: $y'' + 4y = \tan x$

1) Find solution to homogeneous equ $y'' + 4y = 0$
$\Rightarrow$ char equ: $r^2 + 4 = 0$, $r_{1,2} = \pm 2i$
$\Rightarrow y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$

2) $W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$
$= 2(\cos^2 2x + \sin^2 2x) = 2$

3) $u_1' = -\dfrac{\tan x \cdot \sin 2x}{2} = -\dfrac{1}{2} \dfrac{\sin x}{\cos x} \sin 2x = -\sin^2 x$

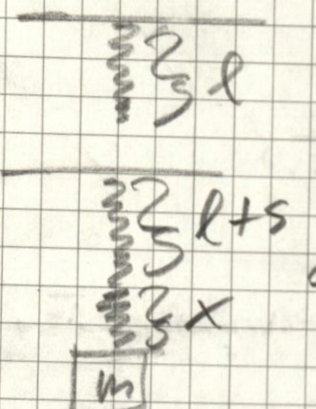
$u_2' = \dfrac{\tan x \cdot \cos 2x}{2} = \dfrac{1}{2} \dfrac{\sin x}{\cos x} (\cos^2 x - \sin^2 x)$
$= \dfrac{1}{2} \left(\sin x \cos x - \dfrac{\sin^3 x}{\cos x}\right)$

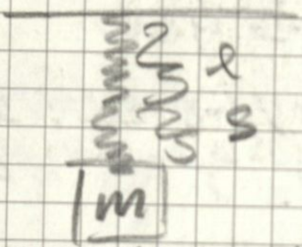
4) $u_1(x) = -\int \sin^2 x \, dx = \int \dfrac{\cos 2x - 1}{2} dx = \dfrac{1}{4} \sin 2x - \dfrac{x}{2}$

$u_2(x) = \dfrac{1}{2} \int \left(\sin x \cos x - \dfrac{\sin x (1 - \cos^2 x)}{\cos x}\right) dx \qquad v = \cos x, \quad dv = -\sin x \, dx$
$= \dfrac{1}{2} \int \left(-v + \dfrac{1 - v^2}{v}\right) dv = -\dfrac{v^2}{4} + \dfrac{1}{2} \ln v = -\dfrac{\cos^2 x}{2} + \dfrac{1}{2} \ln \cos x$

5) $y(x) = c_1 \cos 2x + c_2 \sin 2x + \cos 2x \cdot \left(\dfrac{1}{4} \sin 2x - \dfrac{x}{2}\right)$
$+ \sin 2x \left(\dfrac{1}{2} \ln \cos x - \dfrac{\cos^2 x}{2}\right).$

Oscillations:





equilibrium: $mg = ks$

Hooke's law:

$F = -kS$

↳ coeff of spring

Newton's law:

$mx'' = mg - k(x+s)$

$= -kx + (\cancel{mg - ks})$

$\Rightarrow mx'' = -kx$

$\Leftrightarrow mx'' + kx = 0$

$x'' + \omega^2 x = 0 \Rightarrow$ characterist equ:

$\omega^2 = \frac{k}{m}$

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda^2 = -\omega^2$

$\Rightarrow \lambda_{1,2} = \pm i\omega$

amplitude phase shift

↑ frequency ↑

$\Rightarrow x(t) = c_1 \cos\omega t + c_2 \sin\omega t$

$= \underbrace{\sqrt{c_1^2+c_2^2}}_{A}\left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\phi}\cos\omega t + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\phi}\sin\omega t\right) = A\cos(\omega t - \phi)$, $\phi = \tan^{-1}\frac{c_2}{c_1}$

Lecture Notes: Damping

4.29.24

$mx'' = -kx - \delta x'$, δ: damping coefficient

$\Rightarrow x'' + \underbrace{2\lambda}_{>0} x' + \underbrace{\omega^2}_{>0} x = 0$, $\frac{\delta}{m} = 2\lambda$, $\omega^2 = \frac{k}{m}$

Characteristic Equ: $s^2 + 2\lambda s + \omega^2 = 0$

Zeros: $s_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

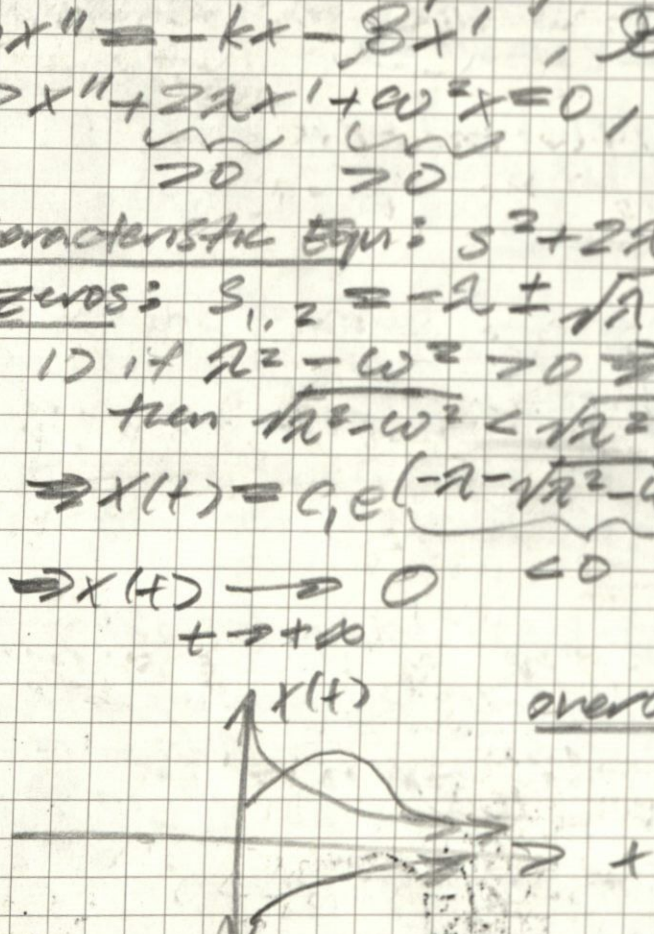
1) if $\lambda^2 - \omega^2 > 0 \Rightarrow s_{1,2} \in \mathbb{R}$

then $\sqrt{\lambda^2-\omega^2} < \sqrt{\lambda^2} = \lambda \Rightarrow s_{1,2} < 0$

$\Rightarrow x(t) = c_1 e^{\overbrace{(-\lambda - \sqrt{\lambda^2-\omega^2})}^{<0} t} + c_2 e^{\overbrace{(-\lambda + \sqrt{\lambda^2-\omega^2})}^{<0} t}$

$\Rightarrow x(t) \xrightarrow[t \to +\infty]{} 0$

overdamping



2) $\lambda^2-\omega^2=0$ critical damping

$\Rightarrow s_1=s_2=-\lambda<0$

$\Rightarrow x(t)=c_1e^{-\lambda t}+c_2te^{-\lambda t}$

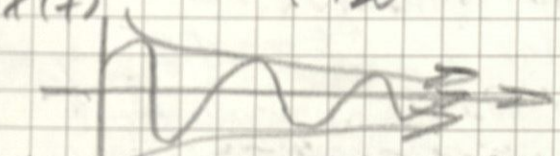
$\Rightarrow x(t)\xrightarrow[t\to+\infty]{}0$ (but not as fast as overdamping)

3) $\lambda^2-\omega^2<0$

$\Rightarrow s_{1,2}=-\lambda\pm i\sqrt{\omega^2-\lambda^2}$

$\Rightarrow x(t)=e^{-\lambda t}(c_1\cos(\sqrt{\omega^2-\lambda^2}\,t)+c_2\sin(\sqrt{\omega^2-\lambda^2}\,t))$

$\Rightarrow x(t)\xrightarrow[t\to+\infty]{}0$



underdamping

Note: $\lim_{t\to+\infty}x_h(t)=0$ fast

$\therefore x(t)\approx x_p(t)$ for large t

Driven or Forced Oscillations:

$x''+2\lambda x'+\omega^2x=F(t)$

$\Rightarrow x(t)=\underbrace{x_h(t)}_{\text{homog. equ}}+\underbrace{x_p(t)}_{\text{particular sol to nonhomog.}}$

transient sol Steady-state sol

Consider Vibration Problem:

$x''+\omega^2x=F_0\sin(\gamma t),\ x(0)=0,\ x'(0)=0$

$\Rightarrow x(t)=\underbrace{c_1\cos\omega t+c_2\sin\omega t}_{x_h(t)}+\underbrace{(A\cos\gamma t+B\sin\gamma t)}_{x_p(t)}$

$x_p''(t)$

$\Rightarrow -\gamma^2(A\cos\gamma t+B\sin\gamma t)+\omega^2(A\cos\gamma t+B\sin\gamma t)=F_0\sin\gamma t$

$\Rightarrow(\omega^2-\gamma^2)A\cos\gamma t+(\omega^2-\gamma^2)B\sin\gamma t=F_0\sin\gamma t$

$\Rightarrow A=0,\ B=\frac{F_0}{\omega^2-\gamma^2}$

$\Rightarrow x(t)=c_1\cos(\omega t)+c_2\sin(\omega t)+\frac{F_0}{\omega^2-\gamma^2}\sin\gamma t$

$x(0)=0=c_1$

$0=x'(0)=\omega c_2+\frac{F_0}{\omega^2-\gamma^2}\gamma\Rightarrow c_2=-\frac{F_0\gamma}{\omega(\omega^2-\gamma^2)}$

Solution to IVP:

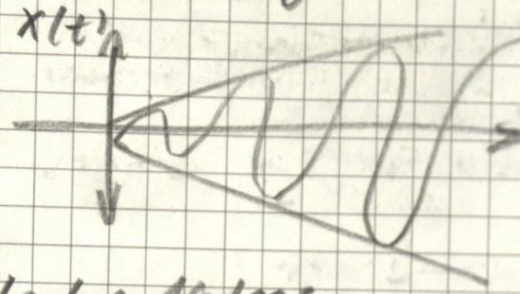
$x(t)=-\frac{F_0\gamma}{\omega(\omega^2-\gamma^2)}+\frac{F_0}{\omega^2-\gamma^2}\sin\gamma t$

$=\frac{F_0}{(\omega^2-\gamma^2)\omega}(\omega\sin\gamma t-\gamma\sin\omega t),\ \gamma\neq\omega$

If $\gamma=\omega$: set $x(t)$ as a quantity dependent on γ and let $\gamma\to\omega$

$\Rightarrow\lim_{\gamma\to\omega}\frac{F_0}{\omega}\frac{\omega\sin\gamma t-\gamma\sin\omega t}{\omega^2-\gamma^2}\underset{\text{L'Hop}}{=}\lim_{\gamma\to\omega}\frac{F_0}{\omega}\frac{\omega t\cos\gamma t-\sin\omega t}{-2\gamma}$

$= \frac{F_0}{\omega}\left(-\frac{\omega t}{2\gamma}\cos\omega t + \frac{\sin\omega t}{2\omega}\right)$



$\Rightarrow +$

Pure Resonance

String will eventually snap!

Abel's Thm 5.1.24

Lecture Notes:

Ex. $y'' - 2ty' + e^t y = 0$

$y_1(0) = 2,\ y_1'(0) = 1$

$y_2(0) = -1,\ y_2'(0) = 3$

Find the Wronskian

Wronskian:

$W(y_1, y_2)(t) = W(y_1, y_2)(0)\, e^{-\int_0^t a_1(s)\,ds}$

$= \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} e^{\int_0^t 2s\,ds} = 7e^{t^2}$

→ and its derivatives

Diff Eq Methods:

1) Substitution: insert function into DE
2) Separation of Variables: $\frac{dy}{dx} = g(y)h(x)$
3) Exact Equations: $f(x,y)dx + g(x,y)dy = 0$ ← implicit form
   s.t. $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow f = \frac{\partial F}{\partial x},\ g = \frac{\partial F}{\partial y} \Rightarrow F(x,y) = C$
4) Integrating Factor: $f(x,y)dx + g(x,y)dy = 0$
   Find $\mu(x,y)$ s.t. $\mu(x,y)f(x,y)dx + \mu(x,y)g(x,y)dy = 0$ is exact.
5) 1st Order Linear ODEs: $y' + a(x)y = b(x)$
6) 2nd Order Linear ODEs:
   a) Constant coefficient, homogeneous
      Solve char. eqn which leads to 3 scenarios:
      1. $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
      2. $c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
      3. $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$
   b) Constant coeff, special RHS: undetermined coefficients
   c) Constant coeff, any RHS: Variation of parameters
   d) Nonconstant coeff, sol y_1 given, find another sol y_2 with reduction of order
      $\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2}$

Ex. Use Variation of Parameters to Solve: $x^2y'' - 3xy' + 4y = x^2\ln x$
knowing $\{y_1, y_2\} = \{x^2, x^2\ln x\}$ is a fund set of sol for homog eqn

$\begin{bmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{x^2\ln x}{x^2} \end{bmatrix} \Rightarrow u_1' = \frac{-(\ln x)^2 x^2}{x^3} = \frac{-(\ln x)^2}{x}$

$\Rightarrow u_1 = \frac{-(\ln x)^3}{3},\ u_2' = \frac{x^2\ln x}{x^3} \Rightarrow u_2 = \frac{1}{2}(\ln x)^2$

$\Rightarrow y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{1}{6}x^2(\ln x)^3$

Independent Notes : Diff Eq Review 5.4.24

1.1, 1.2 : classifications, solutions, IVPs

Ordinary DEs : 1 dependent var, 1 independent var

Partial DEs : ≥ 1 dependent var, ≥ 1 independent var

Order : highest derivative in equation

Linear : $a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_0(x)y = f(x)$

Nonlinear : equation contains anything nonlinear

Consider ODE $F(x, y, y', \dots, y^n) = 0$ $(*)$

Solution : $y(x) : (*) = 0 : \forall x \in I$

General Solution : solution containing constants of integration

IVPs : $(*)$ with conditions $y(x_0) = y_0$, $y'(x_0) = y_1$, $\dots y^{n-1}(x_0) = y_n$ solved for coefficients of GS

2.1, 2.2 : Direction fields, autonomous equations, seperable eqns

Autonomous DEs : Eqn does not explicitly contain ind var

Direction Fields : Model autonomous DEs by plotting equilibrium solutions to ADE and predicting motion of independent var based on sign of ADEs b/w equilibrium sols

Equilibrium Sols : $y(x) = y_0 = C : y' = f(y) \Leftrightarrow f(y_0) = 0$

Seperable DEs : 1st order ODE in form :

$$\frac{dy}{dx} = g(y)h(x) \Rightarrow \int \frac{dy}{g(y)} = \int h(x)dx \Rightarrow G(y) = H(x) + C$$

Integrating Factor : write linear 1st order ODE in standard form

$$a_1(x)y' + a_0(x)y = g(x) \Rightarrow y' + P(x)y = f(x)$$

Find sol $\mu(x)$ to $\mu'(x) = P(x)\mu(x) \Rightarrow \mu(x) = e^{\int P(x)dx}$

Use to solve SDE :

$$(\mu(x)y)' = \mu(x)f(x) \Rightarrow \mu(x)y = \int \mu(x)f(x)dx + C$$

$$\Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)dx + C\right]$$

2.3, 2.4 : Linear 1st order ODEs, Exact Equations

Exact Equations : 1st order ODE in form :

$$(*)\ M(x,y)dx + N(x,y)dy = 0 \ \exists f(x,y) : \frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

$$\Leftrightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \Leftrightarrow f(x,y) = C$$

Test : $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \stackrel{?}{=} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$ Works with other eqn too

Solution : $\frac{\partial f}{\partial x} = M(x,y) \Rightarrow f(x,y) = \int M(x,y)dx + g(y)$

$$\Rightarrow \frac{\partial f}{\partial y} = N(x,y) \Rightarrow g'(y) = C \Rightarrow g(y) = \int C\,dy$$

Integrating Factor - if $(*)$ not exact, multiply by $\mu(x,y)$

$$\mu M dx + \mu N dy = 0 \Rightarrow \frac{\partial \mu M}{\partial y} \stackrel{?}{=} \frac{\partial \mu N}{\partial x}$$

$$\Rightarrow \mu_x N - \mu_y M = \mu(M_y - N_x)$$ PDE, hard, so break into 2 cases

1) $\mu = \mu(x) \Rightarrow \mu_x N = \mu(M_y - N_x)$

2) $\mu = \mu(y) \Rightarrow -\mu_y M = \mu(M_y - N_x)$

Solve whichever eqn has a sol depending on only 1 ind var

2.4, 3.1: More on exact eqns, modeling with 1st order eqns

Growth. $\frac{dM}{dt} = kM$, $M(0) = M_0 \Rightarrow M(t) = M_0 e^{kt}$

Decay. $\frac{da}{dt} = -ka$, $a(0) = a_0 \Rightarrow a(t) = a_0 e^{-kt}$

Logistic. $\frac{dP}{dt} = aP(1 - \frac{b}{a}P) \Rightarrow P(t) = \frac{ac}{bc + e^{-at}}$

⋮

4.1. Second Order Linear ODEs

Independence: Solution set $f_1(x), \dots f_n(x) \in I \in \mathbb{R}$ is lin. dep. $\exists c_1, \dots c_n$ not all 0 s.t. $c_1 f_1(x) + \dots + c_n f_n(x) \equiv 0$ $(\forall x \in I)$

otherwise $f_1(x), \dots f_n(x)$ linearly independent

Superposition: if $y_1, y_2, \dots y_k$ are solutions to linear DE, $\forall c_1, c_2, \dots c_k$ $c_1 y_1 + c_2 y_2 + \dots + c_k y_k$ is also a solution

Check. $\det(A(f_1, f_2, \dots f_n)) \neq 0 \Leftrightarrow f_1, \dots f_n$ lin independent

4.1, 4.2. More on 2nd order linear ODEs, Wronskians, reduction of order

Standard Form: $y'' + a_1(x) y' + a_0(x) y = g(x)$ (*)

Wronskian:

$$W(f_1, \dots f_n) = \begin{vmatrix} f_1 & \cdots & f_n \\ \vdots & & \vdots \\ f_1^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

Fundamental Set of Solutions: $\forall y_1, \dots y_n$ lin independent solutions to homogeneous (*) $(g(x) \equiv 0) \in I$

Reduction of Order:

given y_1 is a sol to homog (*) $\in I$

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2} = \frac{e^{\int -a_1(x)dx}}{y_1^2} \Rightarrow y_2 = y_1(x) \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

4.3. 2nd order homog DEs w/ constant coeff. Char Eqn:

Standard Form: $y'' + ay' + by = 0 \Rightarrow \lambda^2 + a\lambda + b = 0$

Solution. $\lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\lambda_1 = \lambda_2 \in \mathbb{R} \Rightarrow c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

$\lambda_{1,2} = \alpha + i\beta \in \mathbb{C} \Rightarrow e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

4.4. nonhomog DEs – method of undetermined coeff.

General Sol. $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

Trial $y_p(x)$: only works for $g(x) = P(x)$, $P(x)e^{ax}$, $P(x)e^{ax} \sin \beta x$ or $\cos \beta x$

$g(x)$: $P(x) = p_n x^n + \dots + p_1 x + p_0 \Rightarrow y_p(x): x^s(A_n x^n + \dots A_1 x + A_0)$

$P(x)e^{ax} \Rightarrow x^s e^{ax}(A_n x^n + \dots + A_1 x + A_0)$

$P(x) e^{ax} \sin \beta x \Rightarrow x^s e^{ax} P(x) \cos \beta x + x^s e^{ax} \sin \beta x$

Superposition: $y_p(x) = y_{p_1}(x) + \dots + y_{p_k}(x)$

4.6. Variation of Parameters: works for any $g(x)$

Standard Form: $y'' + a_1(x) y' + a_0(x) = g(x)$

Solution: $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$

$$u_1' = \frac{-y_2 g(x)}{W(y_1, y_2)}, \quad u_2' = \frac{y_1 g(x)}{W(y_1, y_2)}$$

$$\Rightarrow y_p(x) = y_1(x) \int \frac{-y_2(x) g(x)}{W(y_1, y_2)} + y_2(x) \int \frac{y_1(x) g(x)}{W(y_1, y_2)}$$

* Review 5.1. Vibrations